Introduction to Superconductivity

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Abstract

An introduction to superconductivity for a graduate course level.

Contents

1 Special Relativity

$$P^\mu = (P^o, \mathbf{P}) = (\frac{E}{c}, \mathbf{P}) \tag{1}$$

$$J^\mu = (J^o, \mathbf{P}) = (\rho c, \mathbf{J}) \tag{2}$$

$$A^\mu = (A^o, \mathbf{A}) = (\phi c, \mathbf{A}) \tag{3}$$

$$\nabla^\mu = (\nabla^o, -\nabla) = (\frac{\partial_t}{c}, \mathbf{J}) \tag{4}$$

1.1 Lorentz Transformations

$$x'_o = \gamma(x_o - \beta x_1) \tag{5}$$

$$x'_1 = \gamma(x_1 - \beta x_o) \tag{6}$$

$$\frac{E'}{c} = \gamma(\frac{E}{c} - \beta P_x) \tag{7}$$

$$P'_x = \gamma(P_x - \beta \frac{E}{c}) \tag{8}$$

$$\rho' c = \gamma(\rho c - \beta J_x) \tag{9}$$

$$J'_x = \gamma(J_x - \beta c \rho) \tag{10}$$

$$\phi' = \gamma(\phi c - \beta A_x) \tag{11}$$

$$A'_x = \gamma(A_x - \beta c \phi) \tag{12}$$

A Lorentz invariant quantity is the proper time:

$$d\tau = dt\sqrt{(1 - \beta^2(t)} = \frac{dt}{\gamma(t)} \tag{13}$$

The invariant interval in the spacetime:

$$ds^2 = c^2 dt^2 - x^2 = x^\mu x_\mu \tag{14}$$

The addition of velocity:

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \frac{\mathbf{v}.\mathbf{u}'}{c^2}} \tag{15}$$

$$u_{\perp} = \frac{u'_{\perp}}{\gamma_v (1 + \frac{\mathbf{v}.\mathbf{u}'}{c^2})} \tag{16}$$

$$\tag{17}$$

The energy-moment relation:

$$P'^{\mu} P'_{\mu} = P^{\mu} P_{\mu} = (mc)^2 \tag{18}$$

$$\frac{E^2}{c^2} - \mathbf{P^2} = (mc)^2 \tag{19}$$

The current conservation:

$$\partial \rho + \nabla j = 0 \rightarrow \partial_{\mu} J^{\mu} = 0 \tag{20}$$

1.2 Rapidity

From

$$ct' = \gamma(ct - \beta x) \tag{21}$$

$$x' = \gamma(x - \beta ct) \tag{22}$$

We make

$$\gamma \rightarrow cosh\zeta$$
$$\beta \rightarrow tanh\zeta$$
$$\beta\gamma \rightarrow sinh\zeta$$

The new Lorentz (Minkowskian) transformations:

$$ct' = ctcosh\zeta - xsinh\zeta \tag{23}$$

$$x' = xcosh\zeta - ctsinh\zeta \tag{24}$$

Where the boost parameter or rapidity is

$$\zeta = \frac{1}{2} ln(\frac{1+\beta}{1-\beta}) \tag{25}$$

When the rapidity is pure imaginary, the Lorentz transformations takes the form of pure rotation (Euclidian), and it becomes a rotation angle.

$$\zeta = i\theta \tag{26}$$

1.3 Particle in E or B

The electromagnetic field strength is:

$$F^{\alpha}_{\beta} = \frac{\partial}{\partial x_\alpha} A_\beta - \frac{\partial}{\partial x_\beta} A_\alpha = \partial^\alpha A_\beta - \partial^\beta A_\alpha \tag{27}$$

The EOM in the covariant form (Lorentz force):

$$\frac{\partial U^\alpha}{\partial \tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta \tag{28}$$

$$\text{Where } U^\alpha = (\gamma c, \gamma u) \tag{29}$$

The sourced Maxwell equations in the covariant form:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{30}$$

The dual field strength:

$$\tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \tag{31}$$

The Bianchi Identity:

$$\partial^\alpha F_{\beta\gamma} + \partial^\beta F_{\gamma\alpha} + \partial^\gamma F_{\alpha\beta} = 0 \tag{32}$$

The transformation of the field strength:

$$F^{\beta\nu} = \frac{\partial x'^\beta}{\partial x^\gamma} \frac{\partial x'^\nu}{\partial x^\alpha} F^{\gamma\alpha} \tag{33}$$

The Boosted **E, B**-Field:

$$\mathbf{E}' = \gamma(\mathbf{E} + (\beta \times \mathbf{B})) - \frac{\gamma^2}{\gamma+1} \beta(\beta.\mathbf{E}) \tag{34}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - (\beta \times \mathbf{E})) - \frac{\gamma^2}{\gamma+1} \beta(\beta.\mathbf{B}) \tag{35}$$

$$\tag{36}$$

Where the inverse transformations, as usual, is found by $\beta \to -\beta$. For a specific transformation corresponding to a boost along x_1 with speed $c\beta$:

$$E'_1 = E_1, B'_1 = B_1, \tag{37}$$

$$E'_2 = \gamma(E_2 + \beta B_3), \tag{38}$$

$$B'_2 = \gamma(B_2 + \beta E_3), \tag{39}$$

$$E'_3 = \gamma(E_3 + \beta B_2), \tag{40}$$

$$B'_3 = \gamma(B_3 + \beta E_2), \tag{41}$$

If there is no B in K, we can see that in the frame K':

$$\mathbf{B}' = -\beta\gamma \times \mathbf{E} \tag{42}$$

If there is no E in K, we can see that in the frame K':

$$\mathbf{E}' = \beta\gamma \times \mathbf{B} \tag{43}$$

Resulting that if either E or B are zero in one system, them in any other system the fields at the point are different of zero.

1.4 Example: Charged Infinite Sheet

E is not perpendicular to the sheet anymore but the Gauss Law still holds.

$$E = 2\pi\sigma \tag{44}$$

$$E'_{\perp} = \gamma E_{\perp} \tag{45}$$

$$E'_{\parallel} = E_{\parallel} \tag{46}$$

$$\sigma' = \gamma\sigma \tag{47}$$

1.5 Example: Proton colliding with Antiproton

A point charge:

$$E = \frac{ex}{(x^2 + y^2 + z^2)^{3/2}} \tag{48}$$

$$E'_r = -\frac{e}{r^2}\frac{1 - \beta^2}{(1 - \beta^2 cos\theta)^{3/2}} \tag{49}$$

The maximum intensity is for $\theta = 0$, and we have:

$$E_r = -\frac{\gamma e}{r^2} = E_{max} \tag{50}$$

The magnetic field is then:

$$B = -\frac{1}{c^2} v \times E \tag{51}$$

1.6 Motion in a Uniform Static B

$$\frac{dp}{dt} = \frac{e}{c} v \times B, \ \frac{dE}{dt} = 0 \tag{52}$$

$$\frac{dv}{dt} = v \times \omega_B \text{ where } \omega_B = \frac{eB}{\gamma mc} = \frac{ecB}{E} \tag{53}$$

1.7 Motion in a Uniform Static B and E

The Lorentz force equation for the particle in K' is:

$$\frac{dp'}{dt'} = e(E' + \frac{v' \times B'}{c}) \tag{54}$$

If v is chosen perpendicular to B and E ($E \times B$ drift):

$$v = c\frac{E \times B}{B^2} \tag{55}$$

And the field in K' is (spiraling lines of force):

$$E'_{\parallel} = 0, B'_{\parallel} = 0 \tag{56}$$

$$E'_{\perp} = \gamma(E + \frac{u}{c} \times B) = 0 \tag{57}$$

$$B'_{\perp} = \frac{B}{\gamma} = (\frac{B^2 - E^2}{B^2})^{1/2}B \tag{58}$$

$B'_{\perp}$ is weaker than B.The drift velocity v has a physical meaning only if $u < c$, i.e. $|E| < |B|$. If $|E| > |B|$, the electric field is so strong that the particle is continually accelerated in the direction of E and its average energy continues to increase with time. One can see it doing a second Lorentz transformation where now the frame K'' moves relative to the first with:

$$v = c\frac{E \times B}{E^2} \tag{59}$$

Now, $B''_{\perp}$ is zero and $E''_{\perp} = \frac{E}{\gamma'}$. In this system, the particle is acted on by a purely electrostatic field which causes hyperbolic motion with ever-increasing velocity.

The fact that a particle can move through crossed E and B fields with velocity $u = cE/B$ provides the possibility of selecting charged particles according to the velocity (in a beam, only particles with this velocity will travel without deflection).

1.8 Solutions of the Maxwell Equations

The inhomogeneous maxwell equations in a covariant form is given by (30):

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\alpha \tag{60}$$

We solve using the Lorentz gauge ($\partial_\alpha A^\alpha = 0$):

$$\Box A^\alpha = \frac{4\pi}{c}J^\alpha(x) \tag{61}$$

$$\text{Where } \Box = \frac{1}{c^2}\partial_t^2 - \nabla^2 \tag{62}$$

The solution of (62) can be found by finding a Green function $D(x,x')$ for the equation:

$$\Box_x D(x,x') = \delta^{(4)}(x-x') \tag{63}$$

In the absence of boundary, the Green function can depend only on the 4-vector difference $z^\alpha = x^\alpha - x'^\alpha$;

$$\Box_z D(z) = \delta^{(4)}(z) \tag{64}$$

Using the Fourier Integrals to transform from coordinate to wave number space:

$$D(z) = \frac{1}{(2\pi)^4}\int d^4k \tilde{D}(k) e^{-ik^\mu z_\mu}. \tag{65}$$

With the definition of

$$\delta^{(4)} = \frac{1}{(2\pi)^4}\int d^4k e^{-k^\mu z_\mu}, \tag{66}$$

we find:

$$\tilde{D}(k) = -\frac{1}{k^\mu k_\mu} \tag{67}$$

and (65) can be written as:

$$D(z) = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-k^\mu z_\mu}}{k^\mu k_\mu}. \tag{68}$$

Performing the contour integrations for the upper contour (a) and the lower contour (r) we find the *advanced Green's function* and the *retarded Green's function*, respectively:

$$D_r(x-x') = \frac{1}{2\pi}\theta(x_0 - x_0')\delta((x-x')^2) \tag{69}$$

$$D_a(x-x') = \frac{1}{2\pi}\theta(x_0' - x_0)\delta((x-x')^2) \tag{70}$$

The solution of the wave equation (62) can be written as:

$$A^\alpha(x) = A^\alpha_{in}(x) + \frac{4\pi}{c}\int d^4x' D_r(x-x') J^\alpha(x') \tag{71}$$

$$A^\alpha(x) = A^\alpha_{out}(x) + \frac{4\pi}{c}\int d^4x' D_a(x-x') J^\alpha(x') \tag{72}$$

Where $A^\alpha_{in}(x)$ and $A^\alpha_{out}(x)$ are the homogeneous solutions. The radiation field are defined as the difference between the outgoing and the incoming fields.

2 Dispersive Media

The Maxwell equations without sources (nonconductor medium):

$$\nabla.B = 0 \tag{73}$$

$$\nabla.D = 0 \tag{74}$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \tag{75}$$

$$\nabla \times H - \frac{\partial D}{\partial t} = 0 \tag{76}$$

Assuming a solution with harmonic time dependence:

$$e^{-i\omega t} \tag{77}$$

The equation of motion become:

$$\nabla.B = 0 \tag{78}$$

$$\nabla.D = 0 \tag{79}$$

$$\nabla \times E = i\omega B \tag{80}$$

$$\nabla \times H = -i\omega D \tag{81}$$

For uniform isotropic linear media $D = \epsilon E$ and $B = \mu H$, the Helmholtz equation:

$$(\nabla^2 + \mu\epsilon\omega^2)(E, B) = 0 \tag{82}$$

The wave number k and the frequency ω are related by $k = \sqrt{\mu\epsilon}\omega$, so we can write the phase velocity as:

$$v = \frac{\omega}{k} = \frac{c}{n} \tag{83}$$

$$n = \frac{c}{v} \tag{84}$$

$$k = \frac{\omega}{c} n \tag{85}$$

For a homogeneous isotropic dielectric with $n(\omega)$, the general solution is:

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} [A(\omega) e^{i(\frac{\omega}{c})n(\omega)x} + B(\omega) e^{-(\frac{\omega}{c})n(\omega)x}] \tag{86}$$

If U(x,t) is real, $n(-\omega) = n^*(\omega)$. If there is a current in the media:

$$\nabla \times H = J + \frac{dD}{dt} = \sigma E + (-i\omega)\epsilon E = (-i\omega)(\epsilon + \frac{i\sigma}{\omega})E \tag{87}$$

$$\sigma(\omega) = \epsilon + \frac{i\sigma}{\omega} \tag{88}$$

$$\frac{dE^2}{dx^2} - \frac{e}{c^2}\frac{d^2E}{dt^2} = \frac{4\pi}{c^2}\sigma\frac{dE}{dt} \tag{89}$$

$$\frac{dH^2}{dx^2} - \frac{e}{c^2}\frac{d^2H}{dt^2} = \frac{4\pi}{c^2}\sigma\frac{dH}{dt} \tag{90}$$

2.1 Retardation Effect

All media shows some dispersion: only over a limited range of frequencies or in vacuum, the velocity of propagation can be treated as constant in **frequency**. The **Drude Model** is an example of how classical matters respond to an external electric field.

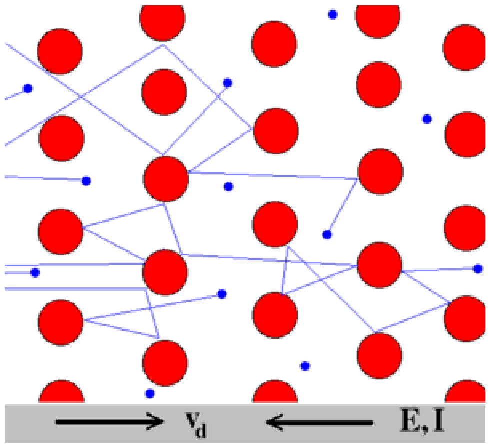

Figure 1: Drude Model electrons (shown here in blue) constantly bounce between heavier, stationary crystal ions (shown in red).

For substances with relativity low density, the relativity permeability is taken as unity. The equation of motion for an electron bound by a harmonic force and acted by an electric field $\mathbf{E(x,t)}$ is:

$$m[\ddot{x} + \gamma\dot{x} + \omega_o^2 x] = -eE(x,t) \tag{91}$$

Where γ is the phenomenological damping. The dipole moment contributed by one electron:

$$p = -ex = \frac{e^2}{m}(\omega_o^2 - \omega^2 - i\omega\gamma)^{-1}E \tag{92}$$

If we suppose N molecules per volume with Z electrons, then the dielectric constant is:

$$\frac{\epsilon(\omega)}{\epsilon_o} = 1 + \frac{Ne^2}{\epsilon_o m}\sum_j f_j(\omega_j^2 - \omega^2 - i\omega\gamma_j)^{-1} \tag{93}$$

$$\epsilon(\omega) = 1 + 4\pi\chi_E(\omega) = 1 + \frac{4\pi Ne^2}{m}\frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega} \tag{94}$$

$$\epsilon_R(\omega) = Re\epsilon_E(\omega) = 1 + \frac{\omega_p^2(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + \gamma^2 + \omega^2} \tag{95}$$

$$\epsilon_I(\omega) = Im\epsilon_E(\omega) = \frac{\omega_p^2(\gamma\omega)}{(\omega_o^2 - \omega^2)^2 + \gamma^2 + \omega^2} \tag{96}$$

To show the anomalous dispersion, we do:

$$\frac{dn}{d\omega}|_{\omega=\omega_k} \approx \frac{2Ne^2\pi}{m}\frac{2\omega_k - i\gamma}{(\gamma\omega_k)^2} < 0 \tag{97}$$

In the anomalous region:

$$v_f = \frac{c}{n(\omega) + \omega\frac{dn}{d\omega}} = \frac{c}{n - \omega^*} > c \tag{98}$$

2.2 Dielectric

$$D(t) = \epsilon(\omega)E(\tau) \tag{99}$$

For an anisotropic media:

$$D(t) = \epsilon(\omega)\delta_{ij}E_i(\tau) \tag{100}$$

The damping constants are small compared with the binding or resonant frequencies, so the dielectric constant is real for most frequencies.

Normal dispersion is associated to an increase in its real part and *anomalous dispersion* to the imaginary part. Normal dispersion occurs everywhere except in the neighborhood of a resonant frequency. A positive imaginary part represents **dissipation of energy** from the wave in the medium, and regions where it is large are in **resonant absorption**.

The attenuation of a plane wave is expressed in terms of real and imaginary parts of the wave number k (where α is the attenuation constant or absorption coefficient):

$$k = \beta + i\frac{\alpha}{2} \tag{101}$$

The intensity of a wave falls off as:

$$e^{-\alpha z}$$

And from (85):

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}, \tag{102}$$

we have:

$$\beta^2 - \frac{\alpha^2}{4} = \frac{\omega^2}{c^2}Re\frac{\epsilon}{\epsilon_o} \tag{103}$$

$$\beta\alpha = \frac{\omega^2}{c^2}Im\frac{\epsilon}{\epsilon_o} \tag{104}$$

if $\alpha \ll \beta$, the attenuation constant can be written as:

$$\alpha \approx \frac{Im\epsilon(\omega)}{Re\epsilon(\omega)}\beta \tag{105}$$

Which means that the fractional decrease in intensity per $\lambda/2\pi$ is $\frac{Im\epsilon}{Re\epsilon}$.

2.3 Conductor - Low Frequency Behavior

In the limit $\omega_0 \to 0$ (static limit),

$$\epsilon(\omega) \approx 1 + \frac{4\pi Ne^2}{m}\frac{1}{-\omega^2 - i\gamma\omega} \tag{106}$$

If some fraction f_o of the electrons per molecules are free, the dielectric constant is singular at $\omega = 0$. Separately:

$$\epsilon(\omega) = \epsilon_b(\omega) + i\frac{Ne^2 f_o}{m\omega(\gamma_0 - i\omega)} \tag{107}$$

Where $\epsilon_b(\omega)$ is the contribution of all the other dipoles. From Maxwell equations and Ohm's law, the harmonic time dependence equation becomes:

$$\nabla \times \mathbf{H} = J + \frac{\partial D}{\partial t} \tag{108}$$

$$\nabla \times \mathbf{H} = -i\omega(\epsilon_b + i\frac{\sigma}{\omega})\mathbf{E} \tag{109}$$

Giving the expression for conductivity (Drude's law):

$$\sigma(\omega) = \frac{f_o Ne^2}{m(\gamma_o - i\omega)} = \frac{Ne^2}{m}\frac{1}{\gamma - i\omega} \tag{110}$$

Where $f_0 N$ is the number of free electrons per volume.

The Drude conductivity is:

$$\sigma(0) = \frac{Ne^2}{m\gamma} \tag{111}$$

The dispersive properties of the medium can be attributed to a complex dielectric constant or to a frequency-dependent conductor plus a dielectric constant.

2.4 Classical Plasma - High Frequency Behavior

We have free charges and at frequencies far above the **highest resonant frequency** the dielectric constant takes the form:

$$\frac{\epsilon(\omega)}{\epsilon_o} \approx 1 - \frac{\omega_p^2}{\omega^2} \tag{112}$$

where the plasma frequency is:

$$\omega_p^2 = \frac{NZe^2}{\epsilon_o m} \tag{113}$$

The dispersion relation (the equation for wave number) is:

$$ck = \sqrt{\omega^2 - \omega_p^2} \tag{114}$$

For $\omega < \omega_p$, k is imaginary, the incident wave in the plasma are deflected and the H-field inside fall off.

For a metal:

$$\epsilon(\omega) \approx \epsilon_b(\omega) - \frac{\omega_p^2}{\omega^2}\epsilon_0 \tag{115}$$

$$\text{Where } \omega_p^2 = \frac{Ne^2}{m^* \epsilon_0} \tag{116}$$

For $\omega \ll \omega_p$, the light incident on the metal has the same behavior than in the plasma.

When $\epsilon(\omega) > 0$ the metal starts to transmit (ultraviolet transparence of metal). The dielectric constant is close to 1 and increases with frequency. The wave number is real.

At $\omega = 0$ the attenuation constant is

$$\alpha_{plasma} \approx \frac{2\omega_p}{c} \tag{117}$$

2.5 Light Waves

For light waves we have the propagation of the pulse in the dispersive media given by:

$$\omega(k) = \frac{ck}{n(k)} \tag{118}$$

The phase velocity is:

$$v_p = \frac{\omega(k)}{k} = \frac{c}{n(k)} \tag{119}$$

The group velocity is:

$$v_g = \frac{c}{n(\omega) + \omega(\frac{dn}{d\omega})} \tag{120}$$

For normal dispersion, $(\frac{dn}{d\omega}) > 0$ and $n > 1$, the velocities are very close. For abnormal dispersion, $(\frac{dn}{d\omega}) < 0$ and the velocities are different.

3 Magnetohydrodynamics Waves and Plasma

The Ohm's law for a fluid in motion is:

$$J = \sigma(E + v \times B) \tag{121}$$

$$\text{In fluid frame } J' = \sigma E' \tag{122}$$

$$\text{In lab frame } J = \sigma(E + \frac{\nabla}{c} \times B) \tag{123}$$

$$\omega \gg \frac{1}{\tau} \tag{124}$$

$$\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m V) = 0 \tag{125}$$

$$\rho_m \frac{\partial V}{\partial t} = -\nabla\rho + \rho_m g + \rho_e(E + \frac{1}{c}V \times B) \tag{126}$$

$$\rho_m \frac{\partial V}{\partial t} = -\nabla\rho + \rho_m g + \frac{1}{c}y \times B + \eta\Delta V \tag{127}$$

It's the case of a non-viscous, perfect conducting ($\nabla^2 B = 0$) fluid. The overall charge is zero. The Drude law:

$$\sigma = \frac{Ne^2}{m}\tau_{collision} \tag{128}$$

$$\tau_{collision} = \frac{m\sigma}{Ne^2} \tag{129}$$

There are two regimes:

- MHD Regime: $\tau_{collision} \ll T = \frac{2\pi}{\omega}$ and $\omega \ll \frac{Ne^2}{m\sigma}$.
- Plasma-wave Regime: $\tau_{collision} \gg T$, with charge separation and $\omega \gg \frac{Ne^2}{m\sigma}$.

3.1 Retardation and Linear Response (Causality)

The linear response is the term ϵ:

$$D(\omega, x) = \epsilon(\omega)E(\omega, x) + O(\epsilon^2) \tag{130}$$

$$D(t, x) = \int \frac{d\omega}{2\pi} e^{-i\omega t} D(\omega, t) \tag{131}$$

$$D(x, t) = \epsilon_0(E(x, t) + \int_{-\infty}^{\infty} G(\tau)E(x, t - \tau)d\tau \tag{132}$$

Observe that equation (132) is non local in time but not in space. Where the retarded response function is:

$$G(t-\tau) = \epsilon(t-\tau) - \delta(t-\tau) \tag{133}$$

$$= \int \frac{d\omega}{2\pi} e^{-i\omega(t-\tau)}(\epsilon(\omega) - 1) \tag{134}$$

A model for $\epsilon(\omega)$ for classical matter is:

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega_o^2 - \omega^2 - i\gamma\omega} \tag{135}$$

$$G(t-\tau) = \int \frac{d\omega}{2\pi} \frac{\omega_p^2}{\omega_o^2 - \omega^2 - i\gamma\omega} e^{-i\omega(t-\tau)} \tag{136}$$

The integral is evaluated by contour integration and have poles in the lower half-plane:

$$\omega_{1,2} = -\frac{i\gamma}{2} \pm \nu_0 \tag{137}$$

By residues, the kernel (136) is:

$$G(\tau) = \omega_p^2 e^{-\gamma\tau/2} \frac{sin\nu_0\tau}{\nu_0} \theta(\tau) \tag{138}$$

Also from (136) the dielectric constant can be expressed in terms of G(τ) as:

$$\epsilon(\omega)/\epsilon_0 = 1 + \int_0^\infty G(\tau) e^{i\omega\tau} d\tau \tag{139}$$

And it shows that the dielectric constant is an analytic function of ω in the upper-half-plane, and G(τ) is finite for all τ. On the real axis it is necessary to require $G(\tau) \to 0$ as $\tau \to 0$.

The analyticity of $G(\tau)$ as $\tau \to \infty$ can also be analyzed by the behavior of $\epsilon(\omega)/\epsilon_0 - 1$ for large ω:

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 \approx \frac{iG(0)}{\omega} - \frac{iG'(0)}{\omega^2} ... \tag{140}$$

The real and imaginary part behave as :

$$Re(\epsilon(\omega) - 1) = O(\frac{1}{\omega^2}) \tag{141}$$

$$Im(\epsilon(\omega)) = O(\frac{1}{\omega^3}) \tag{142}$$

3.2 The Kramers-Kronig Relation

The analytic of $\epsilon(\omega)$ in the upper-half-plane permits the use of Cauchy's theorem to relate the real and the imaginary part of it on the real axis. Cauchy's theorem gives:

$$\frac{\epsilon(z)}{\epsilon_0} = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - z} d\omega' \tag{143}$$

Taking the limit as the complex frequency approaches to real axis ($z = \omega + i\delta$):

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega - i\delta} d\omega' \tag{144}$$

From the relations:

$$\text{Re } z = \frac{z + \bar{z}}{2} \tag{145}$$

$$\text{Im } z = \frac{z - \bar{z}}{2} \tag{146}$$

$$\tag{147}$$

We have the Kramers-Kronig relations (or dispersion relations):

$$\text{Re } \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Im\epsilon(\omega')/\epsilon_0}{\omega' - \omega} d\omega' \tag{148}$$

$$\text{Im } \frac{\epsilon(\omega)}{\epsilon_0} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Re\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega} d\omega' \tag{149}$$

Taking only positive frequencies:

$$\text{Re } \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{2}{\pi}\int_{0}^{\infty} \omega' \frac{\text{Im } \epsilon(\omega')/\epsilon_0}{\omega'^2 - \omega^2} d\omega' \tag{150}$$

$$\text{Im } \frac{\epsilon(\omega)}{\epsilon_0} = -\frac{2\omega}{\pi}\int_{-\infty}^{\infty} \frac{\text{Re } \epsilon(\omega')/\epsilon_0 - 1}{\omega'^2 - \omega^2} d\omega' \tag{151}$$

In **anomalous dispersion** the presence of a very narrow line or band at $\omega = \omega_0$ can be approximate by:

$$Im\epsilon(\omega') \approx \frac{\pi k}{2\omega_0}\delta(\omega' - \omega_0) \tag{152}$$

$$Re\epsilon(\omega') \approx \bar{\epsilon} + \frac{k}{\omega_0^2 - \omega^2} \tag{153}$$

The plasma frequency can be redefined by:

$$\omega_p^2 = \lim_{\omega\to\infty} [\omega^2[1 - \epsilon(\omega)/\epsilon_0]] \tag{154}$$

3.3 MHD and Magnetic Diffusion

The diffusion equation:

$$\frac{\partial B}{\partial t} = \frac{c^2}{4\pi\sigma}\Delta B T_B^2 \tag{155}$$

$$\sigma \to 0, \tag{156}$$

The Typical Magnetic length and time:

$$\frac{1}{L_B} \equiv \frac{c^2}{4\pi\sigma}\frac{1}{T_B^2} \tag{157}$$

3.4 MHD: Magnetic Punch

$$\omega\tau \gg 1 \tag{158}$$

$$\nabla \times B = \frac{4\pi}{y} + \frac{1}{c}\frac{\partial}{\partial t}D \tag{159}$$

(the last part in D is approximately zero).

$$y = \sigma E + \frac{1}{c}(V \times B) \tag{160}$$

$$\nabla \times (\nabla \times B) = \frac{4\pi}{c}(\sigma\nabla \times E + \frac{1}{c}\nabla \times (V \times B)) \tag{161}$$

$$\frac{\partial B}{\partial t} - D\Delta B = \nabla \times (V \times B) \tag{162}$$

$$\rho_m \frac{\partial}{\partial t}V \simeq -\nabla(\rho + \rho_m\phi + \frac{1}{8\pi}B^2) - \frac{1}{4\pi}B(\nabla B) + \eta\Delta V$$

The potential in MHD:

$$U \equiv \rho + \rho_m\phi + \frac{1}{8\pi}B^2 \tag{163}$$

If V is constant, we have the magnetic punch (B creates pressure):

$$\rho_m \frac{\partial}{\partial t}V \simeq -\nabla(\rho + \frac{1}{8\pi}B^2) \tag{164}$$

$$P = -\frac{1}{8\pi}B^2 \tag{165}$$

3.5 Plasma

The charges don't collide on time scale and ions are frozen.

$$\omega\tau_e \gg 1 \tag{166}$$
$$\tau_e = \frac{m\sigma}{Ne^2} \tag{167}$$
$$\sigma = \frac{Ne^2}{m}\tau_e \tag{168}$$

1. $\frac{\partial\rho}{\partial t} + \nabla(\rho V) = 0$
2. $\frac{\partial v}{\partial t} = -\frac{1}{m}\nabla\rho + \frac{e}{m}E + \frac{e}{mc}V \times B + \frac{N}{m}\Delta V$
3. $\nabla E = 4\pi\rho$
4. $\nabla \times B = \frac{4\pi}{c}\rho V + \frac{1}{c}\frac{\partial}{\partial t}E$
5. $Ne_-(V) = Ne^{-\frac{1}{2}\frac{mv^2}{k_b\sigma}} = N_0$
6. $\rho = (N_0 + N)e$, for small distortion we linearize it.
7. $\nabla E = 4\pi(N_0 + N)e = 4\pi Ne$, since overall charge of plasma is zero.

The force for an electronic plasma ($T \approx 0$), where ν is the collision frequency:

$$\frac{dv}{dt} + (v.\nabla)v = \frac{e}{m}(E + \frac{v}{c} \times B) - \nu V \tag{169}$$
$$\frac{dv}{dt} = \frac{eE}{m} - v\omega_p - \nu v \tag{170}$$
$$v = a_0 E_{\parallel} + a_1 E_{\perp} + a_2 E_2 \times \omega_p \tag{171}$$

The Ohm's Law is given then by:

$$J_i = \sum \sigma_{ij} E_j \tag{172}$$

3.5.1 Density Waves

$$\frac{\partial\rho}{\partial t} + \nabla(\rho V) = 0 \tag{173}$$
$$\frac{\partial n_0}{\partial t} + \nabla(n_0 V) = 0, \text{ since } n_0 \text{ is stationary.} \tag{174}$$

For first order we have:

$$\frac{\partial n}{\partial t} + n_0 e \nabla V = 0 \tag{175}$$
$$\frac{\partial^2 ne}{\partial t^2} + n_0 e \nabla(\frac{\partial V}{\partial t}) = 0 \tag{176}$$

$$\rho = (n_o + n)e = (n_o(x) + b(t,x))e \tag{177}$$

$$n_o(v) = -e^{-\frac{mv^2}{2k_bT}} \tag{178}$$

$$1.\ \frac{\partial n}{\partial t} + \nabla(n_o V) = 0 \tag{179}$$

$$2.\ m.n_o\frac{dV}{dt} = -\nabla P - n_o E - n_o\frac{e}{c}V \times B \tag{180}$$

Adding all them up we find:

$$(\frac{\partial^2}{\partial t^2} - c^2\triangle + \omega_p^2)n(t,x) = 0 \tag{181}$$

where

$$c^2 = \frac{2}{m}\frac{\partial P}{\partial n_o} \tag{182}$$

Resulting on:

$$P(n) = P_o(\frac{n}{n_o})^\gamma \text{ where } \gamma = \frac{c_p}{c_v} \tag{183}$$

3.5.2 The Vlasov Equations

The effects of finite temperature on plasma can be described by Vlasov Equations. Letting $f(x,v,t)$ be a distribution function of electrons in the plasma, the equation is"

$$\frac{df}{dt} = \frac{df}{dx} + V\nabla \times f + aV_v f = 0 \tag{184}$$

3.5.3 Example: 1D fluid where $\gamma = 3$

$$\frac{\partial P}{\partial n_o} = 3\frac{P_o}{n_o} = \frac{3n_oKT}{n_o} = 3KT \tag{185}$$

$$c^2 = \frac{1}{m}\frac{\partial P}{\partial n_o} \tag{186}$$

$$c = \sqrt{3}\sqrt{\langle v^2\rangle} = \sqrt{3}v_t \tag{187}$$

3.5.4 Landau Damping

The effect of damping (exponential decrease as a function of time) of longitudinal space charge waves in plasma. This phenomenon prevents an instability from developing, and creates a region of stability in the parameter space.

$$v = \frac{\omega(k)}{|k|} = \frac{1}{|k|}\sqrt{\omega_p^2 + c_L^2 k^2} \equiv \frac{\omega_p}{|k_p|} \tag{188}$$

$$k_p^2 = \frac{\omega_p^2}{\langle v^2 \rangle} = \frac{4\pi N e^2}{m}\frac{1}{\langle v^2 \rangle} = \frac{4\pi N e^2}{k_p \sigma} \tag{189}$$

$$\tag{190}$$

Where we find the thermal velocity:

$$v_p = \frac{\omega_p}{k_p} = \sqrt{\langle v^2 \rangle} \tag{191}$$

4 Superconductivity

Facts:

1. Sustain Persistent Currents (impurities dissipate currents).
2. Meissner effect (type II).
3. Suppression of superconductivity by a strong H-field or high temperature. A superconductor is perfectly diamagnetic, consequence of an effective γ mass.

4.1 The Meissner Effect

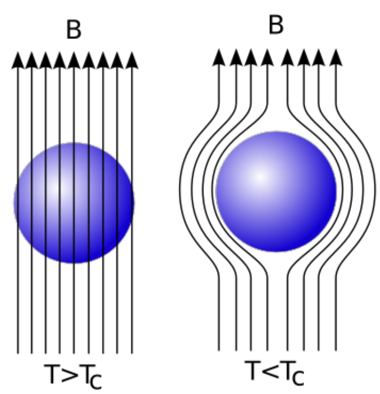

Figure 2: The Meissner Effect

It's the expulsion of **B** from the interior of superconductor as it makes a transition from normal state ($T > T_c$) to superconducting state ($T < T_c$).

4.2 London Equation

Assuming the current flow within a superconductor is caused by non-relativistic motion of charge carrier of charge Q, effective mass m_Q, density N_Q, if the average local velocity of these carriers is V, the current density is :

$$J_s = QN_QV \tag{192}$$

In an electric field E, the current can be expressed in the canonical momentum $P = n_Qv + \frac{QA}{c}$:

$$J = \frac{Q}{m_Q} n_Q P - \frac{Q^2}{m_Q c} A m_Q, \tag{193}$$

The superconducting state is a coherent state of charges carriers with a vanishing canonical momentum (p=0). The effective current is a conductor is

$$J = -\frac{Q^2}{m_Q c} n_Q A, \tag{194}$$

and then one has the current density in the Lorentz-gauge wave such as the Proca $\nabla^2 A - \partial_0^2 A - \mu^2 A = 0$.

In the same fashion, the way the Meissner effect was proved was by London, with solutions $e^{\pm \mu x}$:

$$J_s = e_s N_s V_s \tag{195}$$

The relation with the canonical momentum (no motion in a ground state of a superconductor):

$$\frac{dJ}{dt} = \frac{n_s e^2}{m} E \tag{196}$$

$$\nabla \times J = -\frac{n_s e_s^2}{m_s c} B \tag{197}$$

$$p = mV + \frac{e_s}{c} A \tag{198}$$

$$\langle p \rangle_s = 0 \tag{199}$$

$$m_s \langle V_s \rangle + \frac{e_s}{c} A = 0 \tag{200}$$

$$\mathbf{V_s} = -\frac{\mathbf{e_s}}{\mathbf{m_s c}} \mathbf{A} \tag{201}$$

$$J_s = -\frac{n_s e_s^2}{mc} A \tag{202}$$

Consistency requires that the $\nabla V_s = 0$ (current conservation), ie $\nabla A = 0$. So London argument is only valid for Coulomb Gauge. By the

Ampere's Law you have:

$$\nabla \times H = \frac{4\pi}{c} J_s = (\frac{4\pi N_s e_s^2}{m_s c^2}) A \tag{203}$$

$$\nabla \times \nabla \times H = -(\frac{4\pi N_s e_s^2}{m_s c^2}) H \tag{204}$$

$$\nabla^2 B = \frac{1}{\lambda^2} B \tag{205}$$

The penetration depth, which is the characteristic length over which an external magnetic field is exponentially suppressed:

$$(-\nabla^2 + \frac{1}{\lambda_L^2}) H = 0 \tag{206}$$

$$\frac{1}{\lambda_L^2} = \frac{4\pi N_s e_s^2}{m_s c^2} \tag{207}$$

The effective mass of photon is the $(m_\gamma)_{eff} = \frac{\hbar}{\lambda_L c}$. Plugging the penetration depth in the magnetic field, one has:

$$\text{3d: } H(x) = \frac{e^{-|x|/\lambda_L}}{|x|} \tag{208}$$

$$\text{1d: } H(x) = e^{-x/\lambda_L} \tag{209}$$

$$\text{1d: } B(x) = B_0 e^{-x/\lambda_L} \tag{210}$$

$$\tag{211}$$

References

[1] Classical Electrodynamics, J.D. Jackson (1999).

[2] http://en.wikipedia.org/wiki/Superconductivity.