Maxwell-Chern-Simons Theory

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Introduction

Chern-Simons theory is a (2+1)-dimensional gauge theory differently of the (2+1)-Maxwell theory. Together they can be represented by the action

$$S_{MCS} = \int d^3x \Big[-\frac{1}{4e^2} F_{\mu\nu}F^{\nu\mu} + \kappa \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \Big]. \quad (1)$$

In this paper I discuss basic aspects of the Maxwell-Chern-Simons theory and then find the equations of motion and the spectrum of excitations from the analogy to the Landau levels.

1 Maxwell Gauge Theory

The Maxwell gauge theory is defined in terms of the fundamental gauge field $A_\mu = (A_0, \vec{A})$, and its lagrangian is

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu, \quad (2)$$

where the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the matter current J^μ is conserved $(\partial_\mu J^\mu)$.

Gauge Invariance and Equation of Motions

The Maxwell lagrangian is invariant under gauge transformation

$$A_\mu \to A_\mu + \partial_\mu f,$$

and the derived equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu,$$

are gauge invariants. With source free, it has plane-wave solutions.

In (2+1) Dimensions

In (2+1) dimensions the magnetic field is a pseudo-scalar $B = \epsilon^{ij}\partial_i A_j$ rather than a pseudovector $\vec{B} = \vec{\nabla} \times \vec{A}$ in (3+1) dimensions. The electric field $\vec{E} = -\vec{\nabla} A_0 - \vec{A}$ is a two dimensional vector.

2 Chern-Simons Theory

The Chern-Simons lagrangian is

$$\mathcal{L}_{CS} = \frac{\kappa}{2} e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu. \quad (3)$$

Gauge Invariance and Equations of Motion

Let us vary the equation (3) by a space-time derivative

$$\delta \mathcal{L}_{CM} \sim \partial_\mu (e^{\mu\nu\rho} \partial_\nu A_\rho),$$

clearly if we do not consider boundary terms, the action will be gauge invariant. Straightforward from the Euler-Lagrange equation, we find the equations of motion, also gauge invariant,

$$\frac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} = J^{\mu}. \tag{4}$$

The source free solution reduces to $F_{\mu\nu} = 0$, flat connections. We can show explicitly the current conservation:

$$\begin{aligned} \partial_{\mu}(\frac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}) &= \partial_{\mu}J^{\mu}. \\ \epsilon^{\mu\nu\rho}\partial_{\mu}F_{\nu\rho} &= 0. \end{aligned}$$

Chern-Simons + Matter = Anyons

The matter current in equation (4) can be seen better writing $J^{\mu} = (\rho, \vec{J})$ in terms of components, reveling a tying of flux to charge and the nature of *anyons*.

$\rho = \kappa B$	Charge density is locally proportional to Mag. Field. Magnetic flux is proportional to electric charge.
$J^i = \kappa\epsilon^{ij}E_j$	Charge-Flux relation is preserved under time evolution. Proof: $\dot{\rho} = \kappa\dot{B} = \kappa\epsilon^{ij}\partial_i\dot{A}_j$ $\partial_{\mu}J^{\mu} \to J^i$ $= -\kappa\epsilon^{ij}\dot{A}_j + \epsilon^{ij}\chi = \kappa\epsilon^{ij}E_j$

3 Topologically Massive Gauge Theory

Topological electrodynamics (Chern-Simons charged particle system) is a theory describing an interaction of a U(1) gauge field $A(x,t)$, a vector-valued function on the three-dimensional space, with a charged matter field, characterized by a current $J(x,t)$. When we put together the two theories we get a surprising new form of gauge field mass generation, different of the Higgs mechanism. The Maxwell-Chern-Simons lagrangian is

$$\mathcal{L}_{MCS} = -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_{\mu}\partial_{\nu}A_{\rho}. \tag{5}$$

Equations of Motion

Calculating the Euler-Lagrange equation of motion gives

$$\partial_{\mu}F^{\mu\nu} + \frac{\kappa e^2}{2}\epsilon^{\nu\beta\alpha}F_{\beta\alpha} = 0. \tag{6}$$

Massive Gauge Theory

From dimensional analysis we see that $[e^2] = [m]$ and $[\kappa] = [m^0]$ in (2+1) dimensions. Therefore these equations will describing the propagation of a degree of freedom with mass $m = \kappa e^2$. We see the origin of this mass explicitly rewriting equation (6) in terms of a dual gauge invariant field $\tilde{F}^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$, and them rewriting the equations of motion

$$\Big(\partial^{\nu}\partial_{\nu} + (\kappa e^2)^2\Big)\tilde{F}^{\mu} = 0.$$

4 Mass Spectrum (Excitations)

We shall use the analogy to the classic Landau problem of charge moving in the plane in the presence of an external uniform $\vec{B}$ perpendicular to the plan to find the spectrum of our theory. The quantization of the Landau problem is well understood, consists of equally spaced energy levels (Landau levels) by $\hbar\omega_c$, where $\omega_c = \frac{B}{m}$ is the cyclotron frequency. Each Landau level is infinitely degenerated in the open plane, but for a finite area A the degeneracy is related to the net magnetic flux, $\phi = \frac{BA}{2\pi}$.

Canonical Chern-Simon

Let us quantize the lagrangian. We rewrite equation (5) in a canonical structure

$$\mathcal{L}_{MCS} = \frac{1}{2e^2}E_i^2 - \frac{1}{2e^2}B^2 + \frac{\kappa}{2}\epsilon^{ij}\dot{A}_iA_j + \kappa A_0B. \tag{7}$$

In the $A_0 = 0$ gauge, A_i are the 'coordinates' and we have the momentum fields

$$\Pi^i = \frac{\partial\mathcal{L}_{CSM}}{\partial\dot{A}_i} = \frac{1}{e^2}\dot{A}_i + \frac{\kappa}{2}\epsilon^{ij}A_j.$$

The hamiltonian is

$$\mathcal{H}_{MCS} = \Pi^i \dot{A}_i - \mathcal{L}$$

$$\mathcal{H}_{MCS} = \frac{e^2}{2}\Big(\Pi^i - \frac{\kappa}{2}\epsilon^{ij}A_j\Big)^2 + \frac{1}{2e^2}B^2 + A_0\Big(\partial_i\Pi^i + \kappa B\Big)$$

Then $A_i(\vec{x},t), \Pi^i(\vec{x},t)$ satisfy the canonical equal-time Poisson brackets, which becomes the equal-time canonical commutation,

$$[A^i(\vec{x}), \Pi^j(\vec{y})] = i\delta^{ij}\delta(\vec{x}-\vec{y}). \quad (8)$$

Analogy to the Landau Problem

We consider the long wavelength limit of 7, in which we drop all spatial derivatives,

$$L = \frac{1}{2e^2}\dot{A}_i^2 + \frac{\kappa}{2}\epsilon^{ij}\dot{A}_i A_j.$$

Now, thinking about the non-relativistic charged particle moving in the plane, we have

$$L = \frac{1}{2}m\dot{x}_i^2 + \frac{B}{2}\epsilon^{ij}\dot{x}_i x_j.$$

The momenta is $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{B}{2}\epsilon^{ij}x_j$ and the hamiltonian is

$$H = p_i\dot{x}_i - L = \frac{1}{2m}(p_i - \frac{B}{2}\epsilon^{ij}x_j)^2 = \frac{m}{2}v_i^2.$$

In the quantum level, $[x_i, p_j] = i\delta_{ij}$ implies that the velocities do not commute $[v_i, v_j] = -i\frac{B}{m^2}\epsilon_{ij}$. We now can compare both equations/problems,

Maxwell-Chern-Simons	Landau
e^2	$\frac{1}{m}$
κ	B
$m_{MCS} = \kappa e^2$	$\omega_c = \frac{B}{m}$

Physical masses of the theory appear as physical frequencies of the corresponding quantum mechanic system. The inclusion of the Chern-Simons term in a gauge theory lagrangian is analogous to the inclusion of a Lorentz force in a mechanical system. The Landau system shows how to obtain characteristic frequency without introducing a harmonic binding term (such as in the Higgs mechanics).

Explicitly, a planar quantum mechanical harmonic system with a hamiltonian of the kind $H = \frac{1}{2m}(p^i + \frac{b}{2}\epsilon^{ij}x^j)^2 + \frac{1}{2}m\omega^2\vec{c}^2$ can be separated into two distinct harmonic oscillators of frequency, let us say for our problem,

$$\omega_\pm = \frac{\omega_c}{2}\Big(\sqrt{1+\frac{4\omega^2}{\omega_c^2}} \pm 1\Big),$$

where ω is the harmonic well frequency.

Taking $\omega_c = \kappa e^2$ and $\omega = \sqrt{2}ev$, the characteristic frequencies are exactly mass poles $m_\pm$. In the limit which the cyclotron frequency dominates,

$$\omega_- \to \frac{\omega^2}{\omega_c} = \frac{2v^2}{\kappa} = m_-$$

and

$$\omega_+ \to \infty.$$

Therefore, we have the analogy for our the (2+1) dimensions Maxwell-Chern-Simons, where the gauge field has two massive modes.

References

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