Gravitational Hydrostatics, Part I

In this exercise, you will consider the application of the polytropic equation of state \( P = K \rho^n \) to hydrostatic equilibrium.

The equations governing Newtonian hydrostatic equilibrium are

\[
\frac{dP(r)}{dr} = -\frac{Gm(r) \rho(r)}{r^2}; \quad \frac{dm(r)}{dr} = 4\pi \rho(r) r^2
\]

Let’s express this in terms of dimensionless variables:

\[
r = A \xi, \quad \theta = \left( \frac{\rho}{\rho_c} \right)^{1/n}, \quad A = \left[ (n+1) K \rho_c^{1/n-1} / (4\pi G) \right]^{1/2},
\]

where \( \rho_c \) is the central density. Then the Lane-Emden equation is found:

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.
\]

Boundary conditions for Eq. (1) are \( \theta = 1 \) and \( \theta' = d\theta/d\xi = 0 \) at \( \xi = 0 \). The outer extent of the polytrope is determined by the value of \( \xi, \xi_1 \), at which \( \theta \) has its first zero. The only parameter, for these boundary conditions, is \( n \). Each value of \( n \) gives rise to a different structure. Changes in other quantities, such as \( \rho_c, G, \) or \( K \), only affect the scale.

Explicit Solution of Differential Equations

There are two basic ways in which the Lane-Emden equation may be solved. The first is to use an explicit scheme, which is a direct integration, or shooting method. The second method is to use an implicit scheme, in which the differential equation is solved by finite differencing. We will pursue the latter technique at a later time.

An excellent numerical technique for explicit integration is provided by the Runge-Kutta method. In this exercise, we will use the 4th order scheme described, for example, in *Numerical Recipes*. Write the equations as

\[
\theta' = y, \\
y' = -\frac{2}{\xi} y - \theta^n.
\]

Beginning from the origin \( \xi = 0 \), where \( \theta = 1 \) and \( y = 0 \), one integrates until \( \theta = 0 \) is reached. This point, \( \xi_1 \), is the outer edge of the configuration. We want to know also the value of \( y_1 \) there.
There are two aspects of this problem that require particular care.

1.) At the origin $\xi = 0$ the second of the above equations is apparently singular. Actually, it is not, because $y \rightarrow 0$ “faster” than $\xi$ does. However, the computer doesn’t know this, and it is necessary to expand $\theta(\xi)$ near the origin in a power series to find the $\xi = 0$ limit of $y/\xi$. Demonstrate that for \textit{all} polytropic indices $n$, this series expansion begins

$$\theta = 1 - \xi^2/6 + \cdots.$$  

One sees, therefore, that $y/\xi = -1/3$ near the origin.

2.) In the Runge-Kutta method, as in all explicit schemes, one takes steps of finite size in $\xi$. Near the outer boundary, however, one must take care that a step does not result in $\theta < 0$. The computer will croak if it has to take a negative number to a non-integer power. Show that $\theta'' = y'$ is positive near $\theta = 0$ for all $n$. Also show that a limiting stepsize, guaranteed not to be large enough that $\theta < 0$ in the interval, is given by $\delta \xi = -\theta / \theta' = -\theta / y$. Limiting the stepsize to be smaller than this allows one to gradually approach the boundary. Conitue until the stepsize $\delta \theta$ is very small. The values of $\xi$ and $y$, i.e., $\xi_1$ and $\theta'_1$, are then determined, one may deduce $R$ and $M$ for a given value of $\rho_c$.

As a tuneup, one should begin by solving the case $n = 0$, which has an analytic solution. Once your code works, check the values given in the table for solutions of the Lane-Emden equation for various values of $n$. Finally, establish the solutions for the case $n = 3/7$ which will have importance in a future lecture.