From the Hayashi Track to the Main Sequence:

During collapse, the Virial Theorem says that half the energy goes into radiation and the other half into heat (if $n$ fixed):

$$L = -\frac{1}{2} \frac{dE}{dt} = \frac{n - 3 GM^2}{5 - n} \frac{dR}{dt};$$

$$\frac{dL}{dt} = -\frac{1}{2} \frac{d^2E}{dt^2} = \frac{n - 3 GM^2}{5 - n} \frac{dR}{dt} \left[ \frac{2}{R} \left( \frac{dR}{dt} \right)^2 + \frac{d^2R}{dt^2} \right].$$

Constancy of $T_{eff}$ with $L = 4\pi \sigma R^2 T_{eff}^4$ implies $dL/L = 2dR/R$ and $d^2R/dt^2 = 4(dR/dt)^2/R$. This leads to a solution

$$R = R_o \left( 1 + \frac{6L_oR_o t}{GM^2} \right)^{-1/3},$$

where $R_o$ and $L_o$ refer to the initial radius and luminosity.

Continued collapse to the main sequence involves a change in the polytropic structure from $n = 3/2$ to $n = 3$ (the standard model). If we neglect this, however, and use the hydrostatic condition $d^2I/dt^2 = 0$ where $I \propto MR^2$ is the moment of inertia, one finds that $Rd^2R/dt^2 = -(dR/dt)^2$. This implies that $dL/L = -3dR/R$, which is now positive, and to a solution

$$R = R_o \sqrt{1 - \frac{4R_o L_o t}{GM^2}}.$$

This also implies that $d \ln T_{eff} / d \ln R = -5/4$ and $d \ln L / d \ln T_{eff} = 12/5$. This phase of collapse is known as the Henyey phase, and results in an increase in $L$ and $T_{eff}$ with stellar shrinkage. The stellar radius, for a one solar mass star, must shrink from $R_{min}$ to $R_\odot$, the effective temperature rises from about 3500 K to 5500 K, so $L_{min} \simeq (3500/5500)^{12/5} \simeq 3 L_\odot$ and $R_{min} \simeq (5500/3500)^{4/5} \simeq 0.7 R_\odot$. The slope in the H-R diagram of the Henyey track is 2.4, smaller than that of the main sequence, so the Henyey trajectory will eventually intersect it.
Note from the above that \( L_{\text{min}} \) varies with mass less steeply than does \( L_{\text{ssm}} \). Since we expect that for solar mass stars that \( L_{\text{min}} / L_{\text{ssm}} < 1 \) (although we didn’t get this result in the above), in principle as mass is decreased there will be a point at which \( L_{\text{min}} = L_{\text{ssm}} \). For small enough masses, the Henyey phase essentially disappears, and a lower limit to stellar masses occurs.

**Main Sequence Structure**

The main sequence is defined as those stars that burn hydrogen to helium in their cores. Lower mass stars do this via the pp cycle, and higher mass stars via the CNO cycle. The latter is much more temperature sensitive, and leads to convective cores in massive stars. The adiabatic temperature gradient is

\[
\left. \frac{dT}{dr} \right|_{\text{ad}} = -\frac{2T}{5P} \frac{dP}{dr} = -\frac{8\pi \mu G \rho_o r}{15 \ N_o} \tag{25}
\]

for a perfect gas near the center. Diffusive transport leads to

\[
\left. \frac{dT}{dr} \right|_{\text{rad}} = -\frac{3 \ \kappa \rho}{4acT^3} \frac{L}{4\pi r^2} \simeq -\frac{\kappa_o \rho_c^2 \epsilon_c r}{acT_c^3}, \tag{26}
\]

where we used the Thomsen opacity (valid at high temperatures) and approximated \( L(r) \simeq (4\pi/3)\rho_c \epsilon_c r^3 \). The condition for convective instability is simply \( |dT/dr|_{\text{rad}} > |dT/dr|_{\text{ad}} \), or

\[
\epsilon_c > \frac{8\pi \mu G ac T_c^3}{15 \ N_o \kappa_o \rho_c} \simeq 1.2 \times 10^4 \left( \frac{T_{c,6}}{14} \right)^3 \left( \frac{150 \ \text{g cm}^{-3}}{\rho_c} \right) \text{ erg g}^{-1}\text{s}^{-1}. \tag{27}
\]

For comparison, the pp cycle has an energy generation rate of

\[
\epsilon_{\text{pp}} \simeq 17 \left( \frac{T_6}{14} \right)^4 \left( \frac{\rho}{150 \ \text{g cm}^{-3}} \right) \text{ erg g}^{-1}\text{s}^{-1}, \tag{28}
\]
so the Sun’s core must be radiative. In a massive star, for CNO, we have
\[
\epsilon_{CNO} \approx 4 \times 10^5 \left( \frac{T_6}{25} \right)^{17} \left( \frac{\rho}{150 \text{ g cm}^{-3}} \right) \text{erg g}^{-1}\text{s}^{-1}; \quad (29)
\]
these stars have convective cores.

If energy generation is very temperature sensitive, approximate the core as having a point-like energy source. Thus, for \( r > 0 \), we have \( L(r) = L \). We look for a power-law solution for \( T \) outside the convective core: \( T = br^m \). With the radiative transport equation,

\[
m = -\frac{1}{4} \quad b^4 = \frac{3\kappa_o \rho_c L}{16\pi ac} \quad (30)
\]
if \( \rho \) is constant. The convective core’s radius will be determined by \( |dT/dr|_{rad} = |dT/dr|_{ad} \):

\[
r_{core} = \left( \frac{3\kappa_o L}{16\pi ac} \right)^{1/9} \left( \frac{15N_o}{32\pi \mu G} \right)^{4/9} \rho_c^{-1/3}. \quad (31)
\]
Using the relation \( \rho_c \propto M/R^3 \) we have

\[
r_{core}/R \propto L^{1/9}/M^{1/3} \propto M^{5/18} \quad (32)
\]
using the standard solar model for \( L(M) \). Convective cores increase with stellar mass.