Table 3: Points and Weights for Double-Gauss Quadrature

<table>
<thead>
<tr>
<th>n</th>
<th>$\mu_i$</th>
<th>$a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\pm \frac{1}{2} (1 \pm \frac{1}{\sqrt{3}})$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\pm \frac{1}{2} (1 \pm \sqrt{\frac{3}{7} - \frac{2}{7} \sqrt{\frac{6}{5}}})$</td>
<td>$\frac{1}{4} + \frac{1}{12} \sqrt{\frac{5}{6}}$</td>
</tr>
<tr>
<td></td>
<td>$\pm \frac{1}{2} (1 \pm \sqrt{\frac{3}{7} + \frac{2}{7} \sqrt{\frac{6}{5}}})$</td>
<td>$\frac{1}{4} - \frac{1}{12} \sqrt{\frac{5}{6}}$</td>
</tr>
</tbody>
</table>

Eq. (62). In short, we use the points and weight in Table 3. The double-Gauss quadrature formula achieves 0.6% accuracy even for $n = 4$. For $n = 2$, one can show that

$$I_i (\tau) = \frac{3}{4} F \left[ \tau + Q + \mu_i + \frac{Le^{-k\tau}}{1 + k\mu_i} \right]$$

$$B (\tau) = J (\tau) = \frac{3}{4} F \left[ \tau + Q + Le^{-k\tau} \right]$$

$$k = \sqrt{\frac{1 - a_2}{\mu_2^2} + \frac{1 - a_1}{\mu_1^2}} = 2\sqrt{3}$$

$$Q = \mu_1 + \mu_2 - \frac{1}{k} = 1 - \frac{1}{2\sqrt{3}}$$

$$L = \frac{(1 - k\mu_1)(1 - k\mu_2)}{k} = \frac{\sqrt{3}}{2} - 1.$$  

Note that this gives an exact result for $q(0)$, and a result for $q(\infty) = Q$ in error by only 0.1%. The discrete ordinate method can be generalized to yield the exact solution, but we won’t work it out here.

The Emergent Flux from a Gray Atmosphere

Although in a gray atmosphere the opacity is independent of frequency, the flux dependence on frequency still varies with depth. We have

$$B (\tau) = \frac{\sigma T (\tau)^4}{\pi} = J (\tau), \quad T (\tau)^4 = \frac{3}{4} T_{\text{eff}}^4 [\tau + q (\tau)].$$

From the frequency dependence of the source function, Eq. (46) yields

$$F_{\nu} (\tau) = 2 \int_{\tau}^{\infty} B_\nu [T (t)] E_2 (t - \tau) \, dt - 2 \int_{0}^{\tau} B_\nu [T (t)] E_2 (\tau - t) \, dt,$$
where the Planck function is

\[ B_\nu (T) = \frac{2h\nu^3}{c^2} \left( e^{\nu/kT} - 1 \right)^{-1}. \]

Using the parameter \( \alpha = \nu/kT_{\text{eff}} \), where \( T_{\text{eff}}/T = (3[t + q(t)]/4)^{-1/4} \), the flux is \( F_\alpha(\tau) = F_\nu(\tau) \frac{d\nu}{d\alpha} \):

\[ \frac{F_\alpha(\tau)}{F} = \frac{30\alpha^3}{\pi^4} \left[ \int_\tau^\infty \frac{E_2 (t - \tau) dt}{e^{\alpha T_{\text{eff}}/T} - 1} - \int_0^\tau \frac{E_2 (\tau - t) dt}{e^{\alpha T_{\text{eff}}/T} - 1} \right]. \]

As the figure shows, for \( \tau = 0(2) \), this peaks near \( \alpha = 3(5) \), and the peak value for \( \tau = 2 \) is 25% smaller than for \( \tau = 0 \). The mean photon energy is degraded as they are transferred from the interior to the surface. The Planck function \( (B_\alpha/F) \) for \( T = T_{\text{eff}} \) is shown for comparison: the emergent spectra \( (\tau = 0) \) is slightly harder.

**Correction for Stimulated Emission**

In general, there are 3 types of transitions:

- **Spontaneous Emission**: \( N_{i \rightarrow j} = N_i A_{ij} dt \)
- **(Stimulated) Absorption**: \( N_{j \rightarrow i} = N_j B_{ji} I_{\nu ij} dt \)
- **Stimulated Emission** (enhanced in the presence of a photon of the same energy as the spontaneous transition): \( N_{i \rightarrow j} = N_i B_{ij} I_{\nu ij} dt \)
Note the symmetric process of spontaneous absorption cannot occur. In strict thermal equilibrium, detailed balance occurs, and the photon distribution is the Planck function, so

\[ N_j B_{ji} B_{\nu ij} (T) = N_i \left[ A_{ij} + B_{ij} B_{\nu ij} (T) \right]. \]

The Boltzmann formula must hold for the relative abundances of the two states:

\[ \frac{N_i}{N_j} = \frac{g_i}{g_j} e^{-h\nu_{ij}/kT}. \]

Writing out the Planck function:

\[ A_{ij} \frac{g_i}{g_j} = \frac{2h\nu^3}{c^2} B_{ji} \frac{e^{h\nu_{ij}/kT} - B_{ij} g_i}{B_{ji} g_j} e^{h\nu_{ij}/kT} - 1. \]

The Einstein coefficients are independent of temperature (properties of atoms), which can only happen if

\[ \frac{B_{ij} g_i}{B_{ji} g_j} = 1, \quad A_{ij} = B_{ji} \left( \frac{2h\nu^3 g_j}{c^2 g_i} \right). \]

Now recall from an early discussion that the source function, in the absence of scattering, is the ratio of the emissivity \(j_\nu\) to the opacity \(k_\nu\). The total energy produced per unit volume and flowing through a solid angle \(d\Omega\) is

\[ j_\nu \rho d\nu d\Omega = h\nu N_i (A_{ij} + B_{ij} I_\nu) = N_i A_{ij} h\nu \left( 1 + \frac{I_\nu c^2}{2h\nu^3} \right) \]

and the total absorbed energy is

\[ I_\nu k_\nu \rho d\nu d\Omega = N_j B_{ji} I_\nu h\nu. \]

Then

\[ S_\nu = \frac{j_\nu}{k_\nu} = \frac{N_i A_{ij} \left( 1 + \frac{c^2 I_\nu}{2h\nu^3} \right)}{N_j B_{ji}} = \frac{N_i g_i}{N_j g_j} \left( \frac{2h\nu^3}{c^2} + I_\nu \right). \]

\[ S_\nu = e^{-h\nu/kT} \left( \frac{2h\nu^3}{c^2} + I_\nu \right) = k_\nu B_\nu \left( 1 - e^{-h\nu/kT} \right) + I_\nu e^{-h\nu/kT}. \]
In the equation of radiative transfer, we have 
\[
\frac{dI_\nu}{d\tau} = I_\nu - S_\nu = \left( I_\nu - B_\nu \right) \left( 1 - e^{-h\nu/kT} \right),
\]
which can be turned into 
\[
\frac{dI_\nu}{d\tau} = I_\nu - B_\nu
\]
if the opacity \( \kappa_\nu \) is redefined as \( \kappa_\nu(1 - e^{-h\nu/kT}) \).

**Formation of Spectral Lines**

Definitions:

\[
f_\nu(\mu) = \frac{I_\nu(\mu, 0)}{I_c(\mu, 0)}: \text{ residual intensity}
\]

\[
r_\nu = \frac{F_\nu(0)}{F_c(0)}: \text{ residual flux}
\]

\[
W_\lambda = \int_0^{\infty} (1 - r_\lambda) \, d\lambda: \text{ equivalent width}
\]

The subscript \( \nu \) refers to the line, and the subscript \( c \) refers to the continuum. The equivalent width is the width of a completely black line that absorbs the same number of photons as the spectral line of interest. The integrals range of 0 and \( \infty \) just means “far from the line center”. Note that \( W_\nu \approx (\nu/\lambda)W_\lambda \).

Spectral lines are of two types: pure absorption where the absorbed energy is fully shared with the gas, and resonance lines in which it is not. In the former, the emission of photons is completely uncorrelated with previous absorption. In resonance scattering, the emitted photon is completely correlated with the absorbed photon (coherent scattering). Treating the line and continuum processes separately, the radiative transfer equation is 
\[
\frac{dI_\nu(\mu, \tau_\nu)}{d\tau_\nu} = I_\nu(\mu, \tau_\nu) - \frac{(\kappa + \kappa_\nu) B_\nu + (\sigma + \sigma_\nu) J_\nu}{\kappa + \kappa_\nu + \sigma + \sigma_\nu},
\]
where the optical depth in the line is 
\[
d\tau_\nu = (\kappa + \kappa_\nu + \sigma + \sigma_\nu) \rho \, dz.
\]

**Schuster-Schwarzschild Model**

Suppose we have strong resonance lines formed in a thin layer overlying the photosphere. Then \( \kappa << \sigma \) and \( \sigma_\nu >> \sigma \). Then 
\[
\frac{dI_\nu}{d\tau_\nu} = I_\nu - J_\nu, \quad d\tau_\nu = -\sigma_\nu \rho \, dx.
\]
This looks like the transfer equation for a gray atmosphere, and we must have from radiative equilibrium \( F_\nu (\tau_\nu) = \text{constant} \) for each frequency. From the results for a gray atmosphere, using \( n = 1 \),

\[
I_+ (\tau_\nu) = \frac{3F_\nu (\tau_\nu + 1/\sqrt{3} + Q)}{4}, \quad I_- (\tau_\nu) = \frac{3F_\nu (\tau_\nu - 1/\sqrt{3} + Q)}{4}.
\]

The boundary condition \( I_- (0) = 0 \) implies that \( Q = 1/\sqrt{3} \) and

\[
I_+ (\tau_\nu) = \frac{3F_\nu (\tau_\nu + 2/\sqrt{3})}{4}, \quad I_- (\tau_\nu) = \frac{3F_\nu \tau_\nu}{4}.
\]

If we require that the line intensity on the base of the thin cool gas layer be the same as the emergent intensity of the continuum,

\[
I_+ (\tau_o) = \frac{3F_c (0 + 2/\sqrt{3})}{4} = \frac{3F_\nu (\tau_o + 2/\sqrt{3})}{4}.
\]

The residual flux is just

\[
\tau_\nu = \frac{F_\nu}{F_c} = \left( 1 + \frac{\sqrt{3} \tau_o}{2} \right)^{-1}.
\]

The angular dependence can be found from the classical solution

\[
I_\nu (\mu, 0) = \int_0^\infty \frac{J_\nu (t_\nu) e^{-t_\nu/\mu} dt_\nu}{\mu} + I_c (\mu, 0) e^{-\tau_o/\mu}.
\]

The mean intensity can be approximated as

\[
J_\nu (\tau_\nu) = \frac{1}{2} [I_+ (\tau_\nu) + I_- (\tau_\nu)] = \frac{3F_\nu (\tau_\nu + 1/\sqrt{3})}{4}.
\]

Using this relations, one finds

\[
f_\nu (\mu) = \frac{3F_c}{4I_c (\mu, 0) (1 + \sqrt{3} \tau_o/2)} \left[ \mu + \frac{1}{\sqrt{3}} - \left( \mu + \tau_o + \frac{1}{\sqrt{3}} \right) e^{-\tau_o/\mu} \right] + e^{-\tau_o/\mu}.
\]

In the limit of weak lines, \( \tau_o << 1 \), we find

\[
f_\nu (\mu) \approx 1 - \frac{3F_c}{4I_c (\mu, 0) \tau_o}.
\]
There is no angular dependence except what arises due to limb-darkening of the continuum. Thus scattering lines are visible at all points on the stellar disk with roughly equal strength. In the limit of strong lines, \( \tau_0 \gg 1 \),

\[
f_\nu (\mu) \simeq \frac{\sqrt{3} F_c}{2I_c(\mu, 0) \tau_0} \left( \mu + 1/\sqrt{3} \right).
\]

The range in line strength between the center of the disk and the edge is about 2. This will contrast with that to be found from pure absorption lines, discussed next.

**Milne-Eddington Model**

In the case of pure absorption, we have to specify something about the depth dependence of the opacity and source function, which was unnecessary in the scattering case. Define

\[
\epsilon_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}, \quad \eta_\nu = \frac{\kappa_\nu + \sigma_\nu}{\kappa}, \quad \mathcal{L}_\nu = \frac{\kappa_\nu + \kappa}{\kappa + \kappa_\nu + \sigma_\nu} = \frac{1 + \eta_\nu \epsilon_\nu}{1 + \eta_\nu}.
\]

\( \epsilon_\nu \) measures the importance of absorption to total extinction in the line; \( \eta_\nu \) measures the line strength; \( \mathcal{L}_\nu \) measures net effect of absorption in line and continuum. The line transfer equation is

\[
\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \mathcal{L}_\nu B_\nu - (1 - \mathcal{L}_\nu) J_\nu, \quad d\tau_\nu = (\kappa + \kappa_\nu + \sigma_\nu) \rho dz. \quad (65)
\]

Note in the continuum,

\[
d\tau = \kappa \rho dz = \frac{\kappa d\tau_\nu}{\kappa + \kappa_\nu + \sigma_\nu} = \frac{d\tau_\nu}{1 + \eta_\nu},
\]

so \( \tau = \tau_\nu / (1 + \eta_\nu) \). In the Eddington approximation, \( B(\tau) = a + b\tau \), or

\[
B_\nu (\tau_\nu) = a + b\tau_\nu / (1 + \eta_\nu). \quad (66)
\]

Attempt to solve Eq. (65) by taking moments:

\[
\frac{dF_\nu}{d\tau_\nu} = 4\mathcal{L}_\nu (J_\nu - B_\nu), \quad \frac{dK_\nu}{d\tau_\nu} = \frac{F_\nu}{4}.
\]

With \( K_\nu \approx J_\nu / 3 \),

\[
\frac{d^2 J_\nu}{d\tau_\nu^2} = \frac{3}{4} \frac{dF_\nu}{d\tau_\nu} = 3\mathcal{L}_\nu (J_\nu - B_\nu).
\]
Using the linear relation in Eq. (66), we must have

\[ J_\nu (\tau_\nu) - B_\nu (\tau_\nu) = ce^{-3L_\nu \tau_\nu}, \]

where the positive exponent term vanishes since \( J_\nu \to B_\nu \) as \( \tau_\nu \to \infty \). At the surface, the Eddington approximation leads to \( J_\nu(0) \approx F_\nu(0)/2 \), or

\[ \frac{dJ_\nu}{d\tau_\nu} \bigg|_0 = \frac{3}{4} F_\nu(0) = \frac{3}{2} J_\nu(0) = \frac{3}{2} (a + c) = -c\sqrt{3L_\nu} + \frac{b}{1 + \eta_\nu}. \]

Therefore

\[ c = \left[ \frac{b}{1 + \eta_\nu} - \frac{3}{2} a \right] \left( \sqrt{3L_\nu} + \frac{3}{2} \right)^{-1}, \]

\[ J_\nu (\tau_\nu) = a + \frac{b \tau_\nu}{1 + \eta_\nu} + \frac{b}{1 + \eta_\nu} - \frac{3}{2} a \frac{e^{-3L_\nu \tau_\nu}}{\sqrt{3L_\nu} + \frac{3}{2}}. \]

In the continuum, \( \eta_\nu = 0 \), \( L_\nu = 1 \). The residual flux is then

\[ r_\nu = \frac{F_\nu(0)}{F_c(0)} = \frac{J_\nu(0)}{J_c(0)} = \frac{\left( \frac{b}{1 + \eta_\nu} + a\sqrt{3L_\nu} \right) \left( \sqrt{3} + \frac{3}{2} \right)}{(b + a\sqrt{3}) \left( \sqrt{3L_\nu} + \frac{3}{2} \right)}. \]

The residual intensity requires a specification of the source function,

\[ S_\nu (\tau_\nu) = L_\nu B_\nu (\tau_\nu) + (1 - L_\nu) J_\nu (\tau_\nu). \]

In the continuum \( L = 0 \), so \( S(\tau) = B_\nu(\tau) \).

\[ f_\nu (\mu) = \frac{I_\nu (\mu, 0)}{I_c (\mu, 0)} = \frac{\int_0^\infty S_\nu(t_\nu)e^{-t_\nu/\mu} dt_\nu}{\int_0^\infty S_c(t)e^{-t/\mu} dt} = \frac{a + \frac{b\mu}{1 + \eta_\nu} - \frac{(1 - L_\nu) \left[ \frac{3}{2} a - \frac{b}{\sqrt{3L_\nu} + \frac{3}{2}} \right]}{(a + b\mu) (1 + \mu\sqrt{3L_\nu}) \left( \frac{3}{2} + \sqrt{3L_\nu} \right)}}{(a + b\mu) \left( 1 + \mu\sqrt{3L_\nu} \right) \left( \frac{3}{2} + \sqrt{3L_\nu} \right)}. \] (67)

Note that the second term will vanish in the case of pure absorption (\( L \to 1 \)). In this case

\[ r_\nu = \frac{a\sqrt{3} + \frac{b}{1 + \eta_\nu}}{a\sqrt{3} + b}, \quad f_\nu (\mu) = \frac{a + \frac{b\mu}{1 + \eta_\nu}}{a + b\mu}. \] (68)
In an isothermal atmosphere, \( b = 0 \) and \( r_\nu = f_\nu(\mu) = 1 \), and the line disappears. In the absence of temperature gradients, there can be no spectral absorption lines. Thus, the stronger the source function gradient, the stronger the line. Therefore, late-type stars have stronger features than early-type stars. Late-type stars have visible features at wavelengths shorter than the peak energies, where the spectrum is decaying exponentially. Early-type stars have features at wavelengths longer than the peak energies, on the Rayleigh-Jeans tail where the source function varies more slowly with temperature.

For strong absorption, \( \eta_\nu >> 1 \), we have
\[
  r_\nu = \frac{a\sqrt{3}}{a\sqrt{3} + b}, \quad f_\nu(\mu) = \frac{a}{a + b\mu}, \quad \eta_\nu >> 1
\]

Even the strongest line vanishes as \( \mu \to 0 \) at the limb. For this line of sight, grazing the limb, the effects of temperature gradients are minimized. For weak absorption, \( \eta_\nu << 1 \), we have
\[
  r_\nu = 1 - \frac{b\eta_\nu}{a\sqrt{3} + b}, \quad f_\nu(\mu) = 1 - \frac{b\mu\eta_\nu}{a + b\mu}, \quad \eta_\nu << 1
\]

The line strength is proportional to \( \eta_\nu, \kappa_\nu \), and the number of absorbers.

Now consider the case of pure scattering, \( \epsilon_\nu = 0 \), which requires \( \mathcal{L}_\nu = (1 + \eta_\nu)^{-1} \) as \( \kappa_\nu = 0 \).

For strong scattering,
\[
  r_\nu = \frac{(\sqrt{3} + 2)\sqrt{\mathcal{L}_\nu}}{a\sqrt{3} + b}, \quad f_\nu(\mu) = \frac{a\sqrt{3}\mathcal{L}_\nu(\mu + \frac{2}{3})}{a + b\mu}, \quad \eta_\nu >> 1 \quad (69)
\]

Even if the atmosphere is isothermal, lines will persist to the limb, where the residual intensity is still about \( 1/2 \) of the residual flux. These lines do not depend upon the thermodynamic property of the gas, but upon the existence of a boundary, which permits the selective escape of photons.

For weak scattering, \( \eta_\nu \to 0 \) and \( \mathcal{L}_\nu \to 1 \):
\[
  r_\nu = 1 - \frac{b\eta_\nu}{a\sqrt{3} + b}
  \]
\[
  f_\nu(\mu) = 1 - \frac{b\mu\eta_\nu}{a + b\mu} - \frac{\eta_\nu\left(\frac{3}{2}a - b\right)}{(a + b\mu)(1 + \mu\sqrt{3})\left(\sqrt{3} + \frac{1}{2}\right)}, \quad \eta_\nu << 1 \quad (70)
\]

The residual flux has the same form as for weak pure absorption, but the residual intensity is different: even for an isothermal atmosphere, a weak scattering line will be visible at the limb.