Consider instead conditions near the surface. Generally, there is no incident radiation field. Assuming the emergent intensity nevertheless to be nearly isotropic in the forward direction, we have

\[ J_\nu (0) = \frac{1}{2} \int_0^1 I_\nu (\mu, 0) \, d\mu = \frac{1}{2} \sum_i \frac{I_i (0)}{i + 1} \approx \frac{I_0 (0)}{2}, \]

\[ F_\nu (0) = 2 \int_0^1 I_\nu (\mu, 0) \mu \, d\mu = 2 \sum_i \frac{I_i (0)}{i + 2} \approx I_0 (0). \]

Therefore,

\[ J_\nu (0) = F_\nu (0) / 2, \quad \tau_\nu \to 0 \quad (14) \]

which is the Eddington approximation.

**Solutions of the Radiative Transfer Equation**

**Classical Solution**

For simplicity, we will drop the \( \nu \) subscript (but remember that it is there!).

\[ \mu \frac{dI(\mu, \tau)}{d\tau} = I(\mu, \tau) - S(\mu, \tau) \quad (15) \]

This equation has an integrating factor \( e^{-\tau/\mu} \):

\[ \mu \frac{d}{d\tau} \left[ I(\mu, \tau) e^{-\tau/\mu} \right] = -S(\tau) e^{-\tau/\mu}. \]

Integrating:

\[ I(\mu, \tau) e^{-\tau/\mu} = - \int S(t) e^{-t/\mu} \frac{dt}{\mu}, \quad (16) \]

to within a constant. Suppose we evaluate this between two optical depth points, \( \tau_1 \) and \( \tau_2 \). Then

\[ I(\mu, \tau_1) = I(\mu, \tau_2) e^{(\tau_1-\tau_2)/\mu} + \int_{\tau_1}^{\tau_2} S(t) e^{(\tau_1-t)/\mu} \frac{dt}{\mu}, \quad (17) \]

The emergent intensity of a semi-infinite slab can be found if we take \( \tau_1 = 0 \) and \( \tau_2 = \infty \):

\[ I(\mu, 0) = \int_0^\infty S(t) e^{-t/\mu} \frac{dt}{\mu}, \quad (18) \]
which is a weighted mean of the source function, the weighting function being the fraction of energy that can penetrate from depth \(t\) to the surface. If \(S\) is a linear function of depth \(S(t) = a + bt\) then \(I(\mu, 0)\) is the Laplace transform of \(S\), \(I(0, \mu) = a + b\mu\).

Now suppose that we have a finite atmosphere of thickness \(T\) within which \(S\) is constant. Then the emergent intensity is

\[
I(\mu, 0) = I(\mu, T) e^{-T/\mu} + S \int_0^T e^{-t/\mu} \frac{dt}{\mu}

= I(\mu, T) e^{-T/\mu} + S \left(1 - e^{-T/\mu}\right).
\]

If \(T >> 1\) we have \(I(\mu, 0) = S\): the intensity saturates and is independent of angle.

It is most convenient to discuss this equation at an arbitary point when we impose one of two boundary conditions, either at \(\tau = 0\) or \(\tau = \infty\). If \(\tau_1 = 0\) we have

\[
I(\mu, 0) = I(\mu, \tau) e^{-\tau/\mu} + \int_0^\tau S(t) e^{-t/\mu} \frac{dt}{\mu},
\]

or

\[
I(\mu, \tau) = I(\mu, 0) e^{\tau/\mu} - \int_0^\tau S(t) e^{(\tau-t)/\mu} \frac{dt}{\mu}.
\]

In particular, for \(\mu < 0\) when there is no incident radiation \((I(\mu, 0) = 0)\),

\[
I(\mu, \tau) = - \int_0^\tau S(t) e^{(\tau-t)/\mu} \frac{dt}{\mu}, \quad -1 < \mu < 0
\]

For \(\mu > 0\) we can take \(\tau_1 = \infty\), on the other hand, and using \(\tau_2 = \tau\)

\[
I(\mu, \tau) = \int_\tau^\infty S(t) e^{(\tau-t)/\mu} \frac{dt}{\mu}, \quad 1 > \mu > 0
\]

**Schwarzschild-Milne Integral Equations**

Consider the mean intensity

\[
J(\tau) = \frac{1}{2} \int_{-1}^{+1} I(\mu, \tau) d\mu

= \frac{1}{2} \int_0^1 d\mu \int_\tau^\infty S(t) e^{(\tau-t)/\mu} \frac{dt}{\mu} - \frac{1}{2} \int_0^\tau d\mu \int_0^\tau S(t) e^{(\tau-t)/\mu} \frac{dt}{\mu}.
\]
Interchange the order of integration:

\[
J(\tau) = \frac{1}{2} \int_{\tau}^{\infty} S(t) \, dt \int_{1}^{\infty} e^{-w(t-\tau)} \, dw + \frac{1}{2} \int_{0}^{\tau} S(t) \, dt \int_{1}^{\infty} e^{-w(\tau-t)} \, dw,
\]

where we used \( w = 1/\mu \) in the first term and \( w = -1/\mu \) in the second. The \( w \) integrals are called exponential integrals:

\[
E_n(x) = \int_{1}^{\infty} t^{-n} e^{-xt} \, dt = x^{n-1} \int_{x}^{\infty} t^{-n} e^{-t} \, dt. \tag{22}
\]

Note that

\[
E'_n(x) = -E_{n-1}(x) \tag{23}
\]

and

\[
E_n(x) = \frac{1}{n-1} \left[ e^{-x} - xE_{n-1}(x) \right], \quad n > 1. \tag{24}
\]

For large arguments, an asymptotic expansion exists:

\[
E_1(x) = \frac{e^{-x}}{x} \left[ 1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots \right]. \tag{25}
\]

For small \( x \), we can use

\[
E_1(x) = -\gamma - \ln x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{kk!}, \quad x > 0, \tag{26}
\]

where \( \gamma = 0.5572156 \ldots \) Obviously, \( E_1(x) \) is singular at the origin, but \( E_n(0) = (n-1)^{-1} \) is finite for \( n > 1 \). However, \( E_2(x) \) has a singularity in its first derivative at the origin: \( E'_2(0) = -E_1(0) \).

It is useful to collect also these results for the integrals of elementary functions with \( E_1 \):

\[
\frac{1}{2} \int_{0}^{\infty} E_1|t-\tau|t^{p} \, dt = \frac{p!}{2} \left[ \sum_{k=0}^{p} \frac{\tau^k}{k!} \delta_{\alpha} + (-1)^{p+1} E_{p+2}(\tau) \right], \tag{27}
\]

where \( \delta_{\alpha} = 0 \) if \( \alpha = p + 1 - k \) is even, and \( \delta_{\alpha} = 2/\alpha \) if \( \alpha \) is odd. For \( p = 0 \) \cite{1} \cite{2}, the right-hand side of Eq. (27) is \( 1 - E_2(\tau)/2 \left[ \tau + E_3(\tau)/2 \right] \left( 2/3 - E_4(\tau) + \tau^2 \right) \). Finally, for \( a > 0 \) and \( a \neq 1 \),

\[
\frac{1}{2} \int_{0}^{\infty} E_1|t-\tau|e^{-at} \, dt = \frac{e^{-a\tau}}{2a} \left[ \ln \left| \frac{a+1}{a-1} \right| - E_1(\tau - a\tau) \right] + \frac{E_1(\tau)}{2a}.
\]
The mean flux can be written now as

\[ J(\tau) = \frac{1}{2} \int_{-\infty}^{\tau} S(t) E_1(t - \tau) dt + \frac{1}{2} \int_{-\infty}^{\tau} S(t) E_1(\tau - t) dt \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} S(t) E_1|t - \tau| dt. \quad (28) \]

Similarly, we can find

\[ F(\tau) = 2 \int_{-\infty}^{\tau} S(t) E_2(t - \tau) dt - 2 \int_{-\infty}^{\tau} S(t) E_2(\tau - t) dt, \quad (29) \]

and

\[ K(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} S(t) E_3|t - \tau| dt. \quad (30) \]

Recall that the source function, in the case of coherent isotropic scattering, can be written as

\[ S = \frac{\kappa}{\kappa + \sigma} B + \frac{\sigma}{\kappa + \sigma} J \equiv J + \epsilon(B - J), \quad (31) \]

so we can find an integral equation for the source function itself:

\[ S(\tau) = \epsilon B(\tau) + \frac{1 - \epsilon}{2} \int_{0}^{\infty} S(t) E_1|\tau - t| dt. \quad (32) \]

The Planck function makes this equation inhomogeneous. This equation is more general than the assumptions indicate. As long as the angular dependence of the redistribution function is known, it is possible to do the solid angle integrals and express the source function as moments of the radiation field. The moments can be generated from the classical solution, which yields an integral equation like the above.

Since \( S \) can be written in terms of \( J \), we also have

\[ J(\tau) = \int_{0}^{\infty} \frac{\epsilon}{2} B(t) E_1|\tau - t| dt + \int_{0}^{\infty} \frac{1 - \epsilon}{2} J(t) E_1|\tau - t| dt. \quad (33) \]

Remember that \( \epsilon \) is a function of \( \tau \) and has to be inside the integrals.
Asymptotic Form of the Transfer Equation

The condition of radiative equilibrium demands that at each point
\[ \int_0^\infty (\kappa_\nu + \sigma_\nu) J_\nu (\tau_\nu) d\nu = \int_0^\infty (\kappa_\nu + \sigma_\nu) S_\nu (\tau_\nu) d\nu. \]

For isotropic coherent scattering, we have
\[ \int_0^\infty (\kappa_\nu + \sigma_\nu) J_\nu (\tau_\nu) d\nu = \int_0^\infty \kappa_\nu B_\nu (\tau_\nu) d\nu + \int_0^\infty \sigma_\nu J_\nu (\tau_\nu) d\nu, \]
or simply
\[ \int_0^\infty \kappa_\nu J_\nu (\tau_\nu) d\nu = \int_0^\infty \kappa_\nu B_\nu (\tau_\nu) d\nu. \]  
(34)

The scattering has cancelled out. This suggests that at large optical depth, the source function is nearly the Planck function. Also, the thermal emission is set by the local radiation field.

Now consider great depths in a semi-infinite atmosphere, where we expect that \( S_\nu \approx B_\nu \). Making a Taylor expansion:
\[ S_\nu (t) = \sum_{n=0}^\infty \frac{(t - \tau)^n}{n!} \frac{d^n B_\nu (\tau)}{d\tau^n}. \] 
(35)

Substituting into the classical solution Eq. (16) we find for \( \mu > 0 \)
\[ I_\nu (\mu, \tau) = \sum_{n=0}^\infty \frac{1}{n!} \frac{d^n B_\nu}{d\tau^n} \int_\tau^\infty (t - \tau)^n e^{(\tau-t)/\mu} dt = \sum_{n=0}^\infty \frac{d^n B_\nu}{d\tau^n} \frac{1}{n!} \int_0^\infty x^n e^{-x/\mu} dx \]
\[ = \sum_{n=0}^\infty \mu^n \frac{d^n B_\nu}{d\tau^n} = B_\nu (\tau) + \mu \frac{d B_\nu}{d\tau} + \mu^2 \frac{d^2 B_\nu}{d\tau^2} + \cdots. \] 
(36)

For \( \mu < 1 \), to within terms of \( e^{-\tau/\mu} \ll 1 \), the same result for \( I_\nu (\mu, \tau) \) exists. Therefore
\[ J_\nu (\tau) = \frac{1}{2} \sum_{n=0}^\infty \frac{d^n B_\nu}{d\tau^n} \int_{-\infty}^1 \mu^n d\mu = \sum_{n=0}^\infty \frac{1}{2n+1} \frac{d^{2n} B_\nu}{d\tau^{2n+1}} = B_\nu (\tau) + \frac{1}{3} \frac{d^2 B_\nu}{d\tau^2} + \cdots, \] 
(37)
\[ F_\nu (\tau) = \sum_{n=0}^\infty \frac{4}{2n+3} \frac{d^{2n+1} B_\nu}{d\tau^{2n+1}} = \frac{1}{3} \frac{d B_\nu}{d\tau} + \frac{1}{5} \frac{d^3 B_\nu}{d\tau^3} + \cdots, \] 
(38)
and
\[ K_\nu (\tau) = \sum_{n=0}^{\infty} \frac{1}{2n + 3} \frac{d^{2n} B_\nu}{d\tau^{2n}} = \frac{1}{3} B_\nu (\tau) + \frac{1}{5} \frac{d^2 B_\nu}{d\tau^2} + \cdots. \] (39)

Note the relation to the diffusion approximation Eq. (12) established earlier. We can write
\[ F_\nu = -\frac{4}{3} \left( \frac{1}{\kappa_\nu \rho} \frac{dT}{dr} \right) \frac{dT}{dr}, \] (40)
where the coefficient of $dT/dr$ is the radiative conductivity. This equation is simply the stellar structure luminosity equation established earlier.

It turns out to be conceptually simplifying to keep $\mu$ positive for all rays. Thus, where $\mu < 0$ in the above, we will write $-\mu$ henceforth. The classical solution, using $\tau_1 = 0, \tau_2 = \tau$ and $I_\nu (-\mu, 0) = 0$ for $\mu < 0$, and $\tau_1 = \tau, \tau_2 = \infty$ for $\mu > 0$ is then
\[ I_\nu (+\mu, \tau) = \int_{\tau}^{\infty} \frac{S_\nu (t)}{\mu} e^{(\tau-t)/\mu} dt \quad \mu > 0 \]
\[ I_\nu (-\mu, \tau) = \int_{0}^{\tau} \frac{S_\nu (t)}{\mu} e^{-(\tau-t)/\mu} dt. \quad \mu < 0 \] (41)

**Mean Opacities**

For each a given atmospheric equation, it is possible to write the general frequency-dependent equation in a gray form by defining a different mean opacity. For example, if we wanted a correspondance for the fluxes
\[ \int_{0}^{\infty} \kappa_\nu F_\nu d\nu \equiv \kappa_F F, \]
we could define a flux-weighted mean:
\[ \bar{\kappa}_F = \int_{0}^{\infty} \kappa_\nu (F_\nu / F) d\nu. \] (42)

However, a practical difficulty is that we don’t know $F_\nu$ a priori.

A correspondance for the integrated flux
\[ \int_{0}^{\infty} F_\nu d\nu = F \]
would instead imply the mean \( \bar{\kappa} \):

\[
\frac{1}{\rho \bar{\kappa}} \frac{dK}{dz} = \frac{F}{4} = \int_0^\infty \frac{F_\nu}{4} d\nu = \int_0^\infty \frac{1}{\rho \kappa_\nu} \frac{dK_\nu}{dz} d\nu. \tag{43}
\]

Obviously, we don’t know \( K_\nu \) a priori either, but at great depth, where \( 3K_\nu \rightarrow J_\nu \rightarrow B_\nu \), we can define the Rosseland mean opacity

\[
\frac{1}{\bar{\kappa}_R} = \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dz} d\nu = \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu. \tag{44}
\]

This choice is especially useful, since the frequency-integrated form of the structure equation Eq. (40) would involve precisely this mean.

**Gray Atmospheres**

In a gray atmosphere, there is by definition no frequency dependence. The condition of radiative equilibrium then states simply that

\[
S (\tau) = J (\tau) = B (\tau) = \frac{\sigma T^4}{\pi}, \tag{45}
\]

illustrating that the individual roles of scattering and absorption are irrelevant. The integral equations for the source function and the moments of the radiation field become:

\[
B (\tau) = \frac{1}{2} \int_0^\infty B (t) E_1|t - \tau|dt, \quad J (\tau) = \frac{1}{2} \int_0^\infty J (t) E_1|t - \tau|dt,
\]

\[
F (\tau) = 2 \int_\tau^\infty B (t) E_2 (t - \tau) dt - 2 \int_0^\tau B (t) E_2 (\tau - t) dt,
\]

\[
K (\tau) = \frac{1}{2} \int_0^\infty B (t) E_3|t - \tau|dt. \tag{46}
\]

The zeroth moment of the transfer equation is now

\[
\frac{1}{4} \frac{dF}{d\tau} = J - J = 0 \tag{47}
\]

and the first moment equation is

\[
\frac{dK}{d\tau} = \frac{F}{4}. \tag{48}
\]