The gauge group SU(5) as a simple GUT

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Abstract

The idea of the *Grand Unified Theories* (GUTs) is to embed the *Standard Model* (SM) gauge groups into a large group G and try to interpret the additional resultant symmetries. Currently the most interesting candidates for G are $SU(5)$, $SO(10)$, E_6 and the semi-simple $SU(3) \times SU(3) \times SU(3)$. Since the SM group is rank 4, all G must be at least rank $N - 1 = 4$ and they also must comport complex representations. The $SU(5)$ grand unified model of Georgi and Glashow is the simplest and one of the first attempts in which the SM gauge groups $SU(3) \times SU(2) \times U(1)$ are combined into a single gauge group, $SU(5)$. The Georgi-Glashow model combines leptons and quarks into single *irreducible representations*, therefore they might have interactions that do not conserve the *baryon number*, still conserving the difference between the baryon and the lepton number (B-L). This allows the possibility of proton decay whose rate may be predicted from the dynamics of the model. Experimentally, however, the non-observed proton decay results on contradictions of this simple model, still allowing however supersymmetric extensions of it. In this paper I tried to be very explicit in the derivations of SU(5) as a simple GUT.

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The idea of the *Grand Unified Theories* (GUTs) is to embed the *Standard Model* (SM) gauge groups into a large group G and try to interpret the additional resultant symmetries. Currently the most interesting candidates for G are $SU(5)$, $SO(10)$, E_6 and the semi-simple $SU(3)\times SU(3)\times SU(3)$. Since the SM group is rank 4, all G must be at least rank $N-1=4$ and they also must comport complex representations. The $SU(5)$ grand unified model of Georgi and Glashow is the simplest and one of the first attempts in which the SM gauge groups $SU(3)\times SU(2)\times U(1)$ are combined into a single gauge group, $SU(5)$. The Georgi-Glashow model combines leptons and quarks into single *irreducible representations*, therefore they might have interactions that do not conserve the *baryon number*, still conserving the difference between the baryon and the lepton number (B-L). This allows the possibility of proton decay whose rate may be predicted from the dynamics of the model. Experimentally, however, the non-observed proton decay results on contradictions of this simple model, still allowing however supersymmetric extensions of it. In this paper I tried to be very explicit in the derivations of SU(5) as a simple GUT.

1.1 The Representation of the Standard Model

The current theory of the electroweak and strong interactions is based on the group $SU(3)\times SU(2)\times U_Y(1)$, henceforth called the *Standard Model of Elementary particles*. This theory states that there is a *spontaneous symmetry breaking* (SSB) at around 100 GeV, breaking $SU(2)\times U_Y(1)\rightarrow U_{EM}(1)$ via the *Higgs* mechanism. In the SM, the three generations of quarks are three identical copies of SU(3) triplet and the *right-handed* (RH) antiparticles (or *left-handed* (LH) particles) are SU(2) doublet. The remaining particles are singlet under both symmetries. For the first generation, the RH antiparticles SU(2) doublet are [1,2]

$$\bar{\psi}^{\dagger}=\begin{pmatrix}\bar{u}\\ \bar{d}\end{pmatrix},\bar{l}^{\dagger}=\begin{pmatrix}\bar{e}\\ \bar{\nu}_e\end{pmatrix}.$$

The representation of the RH antiparticle creation operators can be seen in table 1. For the LH particle creation one just takes the complex conjugate

[1]RH neutrinos weren't experimentally observed. It is possible to have only LH neutrinos without RH neutrinos if we could introduce a tiny Majorana coupling for the LH neutrinos.

[2]For the antiparticle of the electron, the positron, for convenience we write in this text $e^+=\bar{e}$.

of RH, where for SU(2), $\bar{2} = 2$, since SU(2) is pseudo-real. The resultant operators are shown in table 2.

Creation Op.	Dim on SU(3)	Dim on SU(2)	Y of U(1)	Representation
$u^\dagger$	Triplet	Singlet	$\frac{2}{3}$	$u^\dagger : (3,1)_{2/3}$
$d^\dagger$	Triplet	Singlet	$-\frac{1}{3}$	$d^\dagger : (3,1)_{-1/3}$
$e^\dagger$	Singlet	Singlet	-1	$e^\dagger : (1,1)_{-1}$
$\bar{\psi}^\dagger$	Triplet	Doublet	$-\frac{1}{6}$	$\bar{\psi}^\dagger : (\bar{3},2)_{-1/6}$
$\bar{l}^\dagger$	Singlet	Doublet	$\frac{1}{2}$	$\bar{l}^\dagger : (1,2)_{1/2}$

Table 1: The representations of the right-handed antiparticle creation operators of the standard model, $SU(3) \times SU(2) \times U(1)$. "Dim" stands for dimension, Y is the hypercharge of the U(1) generators S.

Creation Op.	Dim on SU(3)	Dim on SU(2)	Y of U(1)	Representation
$\bar{u}^\dagger$	Triplet	Singlet	$-\frac{2}{3}$	$\bar{u}^\dagger : (\bar{3},1)_{-2/3}$
$\bar{d}^\dagger$	Triplet	Singlet	$\frac{1}{3}$	$\bar{d}^\dagger : (\bar{3},1)_{1/3}$
$\bar{e}^\dagger$	Singlet	Singlet	1	$\bar{e}^\dagger : (1,1)_{1}$
$\psi^\dagger$	Triplet	Doublet	$\frac{1}{6}$	$\bar{\psi}^\dagger : (3,2)_{1/6}$
$l^\dagger$	Singlet	Doublet	$-\frac{1}{2}$	$l^\dagger : (1,2)_{-1/2}$

Table 2: The representations of the left-handed particle creation operators of the standard model, $SU(3) \times SU(2) \times U(1)$. "Dim" stands for dimension, Y is the hypercharge of the U(1) generators S.

The full $SU(3) \times SU(2) \times U(1)$ RH representation of the creation operators is then

$$u^\dagger \oplus d^\dagger \oplus e^\dagger \oplus \bar{\psi}^\dagger \oplus \bar{l}^\dagger = \tag{1}$$
$$(3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (\bar{3},2)_{-1/6} \oplus (1,2)_{1/2}. \tag{2}$$

The full $SU(3) \times SU(2) \times U(1)$ LH representation of the creation operators is then

$$\bar{u}^\dagger \oplus \bar{d}^\dagger \oplus \bar{e}^\dagger \oplus \psi^\dagger \oplus l^\dagger = \tag{3}$$
$$(\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,1)_{1} \oplus (3,2)_{1/6} \oplus (1,2)_{-1/2}. \tag{4}$$

The standard similar way of writing the representations of (LH) matter as representations of $SU(3) \times SU(2)$ in the SM is

$$(u, d) : (\mathbf{3}, \mathbf{2}); (\nu_e, e^-) : (\mathbf{1}, \mathbf{2}); (u^c, d^c) : (\bar{\mathbf{3}}, \mathbf{2}); (e^+) : (\mathbf{1}, \mathbf{1}). \tag{5}$$

1.2 $SU(5)$ Unification of $SU(3) \times SU(2) \times U(1)$

The breaking of $SU(5)$ into $SU(3) \times SU(2) \times U(1)$ can be done in the same fashion as the breaking of $SU(3)$ into $SU(2) \times U(1)$. The SU(5) breaking occurs when a scalar field (such as the Higgs field) transforming in its adjoint (dimension $N^2 - 1 = 5^2 - 1 = 24$) acquires a *vacuum expectation value* (VEV) proportional to the hypercharge generator,

$$S = \frac{Y}{2} = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix}. \tag{6}$$

These 24 gauge bosons are the double than the usual 12. The additional gauge bosons are called X and Y and they violate the baryon and lepton number and carry both flavor and color. As a consequence, the proton can decay into a positron and a neutral pion[3], with a lifetimes given by

$$\tau_p \sim \frac{1}{\alpha^2_{su(5)}} \frac{M_X^4}{m_p^5}.$$

The $SU(5)$ is then spontaneously broken to subgroups of $SU(5)$ plus $U(1)$ (the abelian group representing the phase from (6)). This SSB can be represented as $\bar{5} \oplus 10 \oplus 1$ to LH particles and $5 \oplus \bar{1}0 \oplus 1$ to RH antiparticles, as we will prove in the following sections.

Unbroken SU(5)

The fundamental representation of $SU(5)$, let us say the vectorial representation $|\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5\rangle$, has dimension N=5[4]. The group is complex having an anti-fundamental representation (complex conjugate representation) $\bar{5}$. The embedding of the standard gauge groups in $SU(5)$ consists in finding a

[3] As we see on the last section of the text, no such decay was observed.

[4] See construction of the Lie groups on reference [1].

$SU(3)\times SU(2)\times U(1)$ subgroup of $SU(5)$. We first look to **5** and try to fit a 5-dimentional subset of (2) on it (and the left-handed (4) on $\bar{5}$). There is two possibilities on (2) that sum up 5 dimensions:

$$(3,1)_{-1/3}\oplus(1,2)_{1/2}, \tag{7}$$

and

$$(3,1)_{2/3}\oplus(1,2)_{1/2}.$$

The second one is not allowed because S, the generator of $U(1)$, will not be traceless[5], therefore this group can not be embed in $SU(5)$ in this case. To verify it just do $\frac{2}{3}\times 3+\frac{1}{2}\times 2\neq 0$ (not traceless), differently of $-\frac{1}{3}\times 3+\frac{1}{2}\times 2=0$ (traceless).

The first possibility, (7) is allowed and represents the embedding. The $SU(3)$ generators, T_a, act on the first indices of the fundamental rep of $SU(5)$ $|\epsilon_1\epsilon_2\epsilon_3 00\rangle$ and the $SU(2)$ generators, R_a, act on the last two $|000\epsilon_4\epsilon_5\rangle$. The $U(1)$ generator, S, obviously commutes with the other generators. The embedding of the first RH subset of fermions on $SU(5)$ is then characterized by the traceless generators

$$\begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix}_{5\times 5}, \begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix}_{5\times 5}, \begin{pmatrix} -\frac{I}{3} & 0 \\ 0 & \frac{I}{2} \end{pmatrix}_{5\times 5}.$$

In the same logic, the embedding of the LH subset of fermions on $SU(5)$ is characterized by the traceless generators

$$\begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix}_{5\times 5}, \begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix}_{5\times 5}, \begin{pmatrix} \frac{I}{3} & 0 \\ 0 & -\frac{I}{2} \end{pmatrix}_{5\times 5}.$$

ψ_i	$5\to(3,1)_{-1/3}\oplus(1,2)_{1/2}$	$u^c,\bar{l}(e^c,\nu_e^c)$
ψ^i	$\bar{5}\to(\bar{3},1)_{1/3}\oplus(1,2)_{-1/2}$	$d^c,l(e,\nu_e)$

Table 3: The $SU(3)\times SU(2)\times U(1)$ embedding on the fundamental representation of $SU(5)$. The index c indicates charge conjugation.

[5] All generators of $SU(N)$ are traceless for definition and construction.

A possible representation of the LH $\bar{5}$, rewriting $|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle$[6] is[7]

$$\begin{pmatrix} d^c_{red} \\ d^c_{blue} \\ d^c_{green} \\ e \\ \nu_e \end{pmatrix}_c = \begin{pmatrix} d^c_3 \\ l_2 \end{pmatrix}_5,$$

and as the RH, given by **5**,

$$\begin{pmatrix} u^c_{red} \\ u^c_{blue} \\ u^c_{green} \\ \bar{e} \\ \bar{\nu}_e \end{pmatrix}_c = \begin{pmatrix} u^c_3 \\ \bar{l}_2 \end{pmatrix}_5.$$

The next representation on $SU(5)$ we will use is the antisymmetric in two indices, **10** and its conjugate $\bar{10}$. The remaining ten-dimensional part of (2) and (4) fits respectively on $\bar{10}$ and **10**. To see how it happens, we observe that $\bar{10} = \bar{5} \otimes_A \bar{5}$ so we can multiply the 5-dimensional subsets on table 3 to form these representations. For instance, the LH particles (4) subset forms

$$\begin{aligned} \left[(\bar{3},1)_{\frac{1}{3}} \oplus (1,2)_{-\frac{1}{2}}\right] \otimes_A \left[(\bar{3},1)_{\frac{1}{3}} \oplus (1,2)_{-\frac{1}{2}}\right] &= (6,1)_{\frac{2}{3}} \oplus (-3,1)_{\frac{2}{3}} \oplus (\bar{3},2)_{-\frac{1}{6}} \oplus (1,3)_{-1} \oplus (1,-2)_{-1}, \\ &= (3,1)_{\frac{2}{3}} \oplus (\bar{3},2)_{-\frac{1}{6}} \oplus (1,1)_{-1}. \end{aligned}$$

where we have used $3 \otimes 3 = \bar{3} \oplus 6$ and $2 \otimes 2 = 1 \oplus 3$ and $\otimes_A$ is the antisymmetric product (the first of each part).

$10 \to (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3}$	q, u^c, e^c
$\bar{10} \to (\bar{3},2)_{-1/6} \oplus (\bar{\bar{3}},1)_{2/3}$	q, d^c, e

Table 4: The $SU(3) \times SU(2) \times U(1)$ embedding on the anti-symmetric representation **10** of SU(5).

Finally, all the remaining fermions transforming as a singlet (**1**) under $SU(5)$[8]. The final embedding of fermions of the standard model into the gauge group $SU(5)$ is shown on table 5.

[6] The index c indicates charge conjugation.
[7] Here we ignore the Cabibo type mixing.
[8] This is necessary because of the evidence for neutrino oscillations.

	$SU(5)$ Decomposition	Fermions	Similar Notation
ψ_i ψ^{ij} •	$5 \rightarrow (3,1)_{-1/3} \oplus (1,2)_{1/2}$ $\bar{1}0 \rightarrow (\bar{3},2)_{-1/6} \oplus (\bar{3},1)_{2/3}$ $1 \rightarrow (1,1)_0$	$u^c, \bar{l}(e^c, \nu_e^c)$ u, d^c, e ν^c	(3,1,-1/3) + (1,2,1/2) (3,1,2/3) + (3,2,-1/6)+(1,1,1)
ψ^i ψ_{ij} •	$\bar{5} \rightarrow (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2}$ $10 \rightarrow (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3}$ $1 \rightarrow (1,1)_0$	$d^c, l(e, \nu_e)$ d, u^c, e^c ν^c	($\bar{3}$,1,1/3) + (1,2,-1/2) ($\bar{3}$,1,-2/3) + (3,2,1/6)+(1,1,1)

Table 5: The $\bar{5} \oplus 10 \oplus 1$, LH particles, and $5 \oplus \bar{1}0 \oplus 1$, RH antiparticles, embedding of SM on $SU(5)$.

Breaking SU(5)

The gauge *bosons* of the model are given by the adjoint **24** of $SU(5)$, transforming as

$$24 \rightarrow (8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0 \oplus (3,2)_{-5/6} \oplus (\bar{3},2)_{5/6}, \tag{8}$$

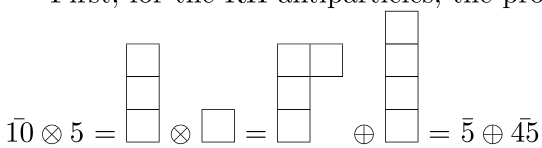
described in detail in table 6.

		SM GB	Add. $SU(3)$	Add. $SU(2)$	Identification
$(8,1)_0$	(8,1,0)	X	-	-	G^α_β
$(1,3)_0$	(1,3,0)	X	-	-	$W^\pm, W^0$
$(1,1)_0$	(1,1,0)	X	-	-	B
$(3,2)_{-5/6}$	(3,2,-5/6)	-	Triplet	Doublet	$A^\tau_\alpha = (X_\alpha, Y_\alpha)$
$(\bar{3},2)_{5/6}$	($\bar{3}$,2,5/6)	-	Triplet	Doublet	$A^\alpha_\tau = (X_\alpha, Y_\alpha)^T$

Table 6: The gauge bosons of SM fitting on the adjoint representation of $SU(5)$. SM stands for standard model, "Add" stands for additional, and GB for gauge boson.

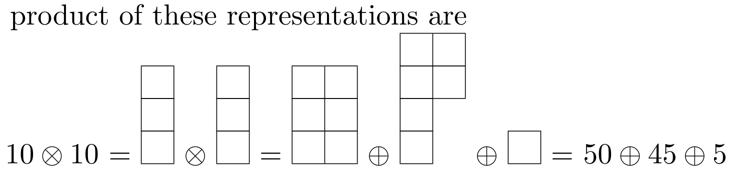
As we already mentioned, the fermions have to acquire mass in $SU(5)$ by a SSB, which also must happen in the GUT theory. For making this to happen, the **product** of the representations containing the fermions and antifermions (table 5), must contain a component which transforms as the $SU(3)\times SU(2)\times U(1)$ Higgs field, represented by $(1,2)_{1/2}$ and $(1,2)_{-1/2}$. It is easy to understand it because the particles and antiparticles of the fermions appears in both **5,** $\bar{10}$ for RH (and **10,** $\bar{5}$ for LH).

First, for the RH antiparticles, the product of these representations are



$$\bar{10}\otimes 5 = \bar{5}\oplus \bar{45}$$

From (7) we see that **5** contains $(1,2)_{1/2}$, it is also contained on **45**, therefore both representations can give mass to $d,\bar{e}$. For the LH particles, the product of these representations are



$$10\otimes 10 = 50\oplus 45\oplus 5.$$

Again **5** and **45** contains $(1,2)_{-1/2}$, giving mass to u, however **50** does not contain $(1,2)_{-1/2}$.

1.3 Anomalies

If the creation generator for all the right hand operators of spin-1/2 particles transform according to a representation generated by T_a^R, we nee to have

$$\text{tr}\left[\{T_R^a,T_R^b\}T_R^c\right]=0. \tag{9}$$

In fact it is true for all simple Lie algebra with exception of SU(N) for $N\leq 3$. Therefore, the *anomaly* in any representation of $SU(N)$ is proportional to

$$\mathcal{D}^{abc}=\text{ tr}\left[\{T_R^a,T_R^b\}T_R^c\right]=\frac{1}{2}A(R)d^{abc}, \tag{10}$$

where

$$2d^{abc}L^c=\{L^a,L^b\}. \tag{11}$$

$A(R)$ is independent of the generators allowing us to choose one generator and calculate it. In our case it is useful to use the generator as the charge operator

$$Q=R_3+S, \tag{12}$$

and looking at (6), one has

$$\frac{A(\bar{5})}{A(10)} = \frac{\text{tr } Q^3(\psi_i)}{\text{tr } Q^3(\psi^{ii})} = -1. \tag{13}$$

It is clear now that the fermions in these representations have their anomalies canceled

$$A(\bar{5}) + A(10) = 0. \tag{14}$$

1.4 Physical Consequences of using $SU(5)$ as a GUT Theory

- The charge of the quarks can be deduced from the fact that there are three color states and from the fact that the charge operators is (12), $Q = R_3 + S = I_3 + \frac{Y}{2}$, and it must be traceless. The multiplet of the **5** representation gives

  $$Q(\nu_e) + Q(\bar{e}) + 3Q(d) = 0 \rightarrow Q(d) = -\frac{1}{3}Q(\bar{e}),$$

  which gives an answer to the charge quantization, not explained n the SM.

- The *Weinberg angle*,

  $$\sin{}^2(\theta_W) = \frac{g'^2}{g^2 + g'^2},$$

  g, g' the coupling constants of gauge bosons in the electroweak theory, cannot be calculated on SM (it is a free parameter). In $SU(5)$ GUT, however, the Weinberg angle is accurately predicted, giving $\sin^2\theta_W \sim 0.21$.

- If a group is simple then its GUT has only one coupling constant before SSB. The three coupling constants of the SM are energy dependent and in $SU(5)$ they unit at $\sim 10^{15}$ GeV. However, a supersymmetric $SU(5)$ is needed to get an exact unification in a single point. Remember that in the SM the strong, weak and electromagnetic fine structure constants are not related in any fundamental way.

- No proton decay was observed, which is a contradiction to its lifetime estimated in $SU(5)$, (7). In a supersymmetric SU(5) the proton lifetime is longer, being apparently experimentally consistent.

- Finally, it is actually clear that the $SU(5)$ might be incomplete when one considers the fact that neutrinos were observed to carry small masses and there might exist extra RH Majorana neutrinos. As we just learned, it is not possible to introduce RH neutrinos trivially in this simple model. One solution is going to the next (complex) gauge group $SO(10)$, where the spinor representation can accommodate sixteen LH fields, or even to E_6, which motivates string theories.

List of Tables

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