Searching for Dark Matter

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Abstract

This is a non-concluded study of possible **Dark Matter signatures** on colliders and astrophysical experiments, together with a initial research on models for the dark sector. The dark sector is an example of a **Hidden Valley** to be searched at LHC with properties motivated and constrained by astrophysical data. This work is mainly divided in three parts: **1)** calculations of the Feynman diagram for light abelian hidden sectors, **2)** calculations of Sommerfeld Enhancement and discussion on many coupling models and for a non-abelian G_{DARK}, and **3)** calculation of a possible Feynman diagram for missing E_T signatures at LHC. To respect the logic of the assignment, I tried to keep the qualitative review in the appendix and the explicitly calculation in the body of the paper. Finally, I apologize for the fact that it became a (too) long review and I hope my diagrams and lagrangians are right.

Contents

1 Hidden Valleys at the LHC

The most expected new Physics at the TeV scale are:

- Supersymmetry.
- Extra Dimensions.
- Technicolor.
- Little Higgs.
- Solution for the hierarchy problem (new heavy states such as: Z', fourth generation, leptonquarks, color octet).

The unexpected signatures at the LHC can come from many classes of models, such as **Hidden Valleys**, **Quirks**, and **Unparticles**, and all these class have implications for dark matter. The dynamics could be give by a hidden sector with new dark forces, which couple only to states in the hidden sector. This hidden sector is neutral under the SM and it can have a connector sector charged under both SM and the hidden sector.

The *AdS/CFT correspondence* states that in certain QCD-like limits (conformal in the ultraviolet), theories are *dual* to a *warped extra dimension*, thus all models can be extended to models which are string motivated, such as a hidden sector in an extra dimensional warped throat. Examples of sectors in the hidden valley are for instance:

- QCD theories with F flavors, N colors.
- QCD theories with only heavy quarks (Quirk limit).
- Pure glue theory.
- N=4 SUSY conformal theory.
- Randall-Sundrum or Klebanov-Strassler.
- Partially Higgsed SU(N).
- Banks-Zaks infrared fixed point (unparticle limit).

These sectors may be organize in terms of α (the gauging coupling) and β (the running of the coupling). The introduction of low mass hidden sectors can be compared to low mass dark matter sectors. On the other hand, the phenomenology of the hidden valleys will be strongly determined by the number of hidden sector *v-quarks*, and whether they are lighter/heavier than the confinement scale of the strong gauge group in the hidden sector.

1.1 Models of Hidden Sector DM

The dark sector may have complex dynamics: it might not have a single stable weakly-interacting particle. The presence of hidden valleys may cause the lightest supersymmetric particle to be unstable and to decay to hidden sector particles, going to the low mass hidden sector. Apart of any new model, dark matter is believed to be:

SINGLE If there is only one new symmetry (R-parity),it results on only one new state. However, at the hidden sectors, there may be other symmetries.

STABLE DM exists in the universe (it is at least long lived).

WEAKLY INTERACTING If it was stronger, it would already been detected in experiments. However, DM may not be weakly interacting with itself.

ELECTRICALLY NEUTRAL With associating strong dynamics, the neutral dark bound state might be charged.

WEAK SCALE It is suggested by the thermal freeze-out calculation that the weak scales gives the observed relic density. However, Hidden Valleys models can given dark matter much lighter without ruins it.

1.2 Light Abelian Hidden Sectors

Hidden valleys might contain low mass hidden sector which can fit dark matter coupling to the SM weakly. This is the same model that would fit the observed radiation excess at INTEGRAL, and the dark matter field can be scalar or fermion together with the gauged MeV mediator U(1), as at figure 1. Assuming that this is a massive gauge mediator, the Feynman diagram considering dark matter (X) as a scalar:

$$(-g_x\gamma^\mu)\Big(\frac{-ig_{\mu\nu}}{((p_1+p_2)^2-M^2}\Big)\bar{u}_{e^-}(-g_e\gamma^\nu)v_{e^+}.$$

The Feynman diagram considering dark matter (X) as a fermion:

$$u_{X^-}(-g_x\gamma^\mu)\bar{v}_{X^+}\Big(\frac{-ig_{\mu\nu}}{((p_1+p_2)^2-M^2}\Big)\bar{u}_{e^-}(-g_e\gamma^\nu)v_{e^+}.$$

In according to [3], the soft mass generated for states in the hidden sector are of the size:

$$m_x^2 = q_x^2 q_I^2 \left(\frac{g_x}{g_e}\right)^2 m_E^2, \tag{1.1}$$

where one can see at the diagram that g_e, g_x are the couplings of the dark sector, q_x, q_t the charges, and m_E the SUSY breaking mass of the right-handed *selectron*.

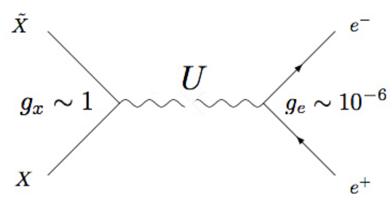

Figure 1: A MeV Dark matter model, where X(scalar or fermion) is the DM field, the gauge mediator is U.

The dark sector and electrons can have a *different coupling to the mediator* through *kinetic mixing* between hypercharge and new hidden U(1), such as $g_e = g_Y \epsilon$, with g_Y the hypercharge gauge coupling, and ϵ the coefficient of the kinetic mixing term. The lagrangian would be then then

$$\mathcal{L}_{kin} = \epsilon F_{\mu\nu} \tilde{F}^{\mu\nu}. \tag{1.2}$$

2 A theory of Dark Matter [1]

The WIMPs can be seen as charged under a hidden gauge symmetry G_{DARK}, broken near to GeV scale. It can be interpreted at INTEGRAL and DAMA as exciting (XDM) or inelastic (iDM). This theory can predict that G_{DARK} particles may be produced in cascade decays of MSSM superpatners at LHC. This will decay in the *Lightest Supersymmetric Particle* (LSP) and other dark particles. Thus, the lightest GeV dark Higgs and gauge bosons will decay into light SM states, mainly leptons (lepton jets). If DM is directly charged to SM, the gauge coupling will give a new long-lived colored particles.

2.1 New Forces in the Dark Sector

- A new force-carrier boson can distort the wavefunction of the incoming particles away from the plane-wave approximation, yielding enhancements/ suppressions to the annihilation cross-section. It can only arise if $m_\phi < \alpha M_{DM} \sim$ **few Ge**, and it is compatible with the mass scale necessary to the ATIC and the large σ by ATIC and PAMELA.

- The possible new annihilation channel is $\chi\chi \to \phi\phi$, which can be dominant.

- A decay into heavier hadronic states/ scalars lighter than $\sim$ 250 MeV and vectors lighter than $\sim$ GeV are forbidden, providing a mode by which dark matter can dominantly annihilate into very hard leptons, with few or no π^0's or antiprotons. If the boson has a small mixing with the standard model, it mass scale can avoid the decaying via hadronic shower, preferring μ, e, $\pi^{\pm}$, instead of π^0's or antiprotons.

- If dark matter interacts with itself via a force carrier with mass $m_\phi \sim GeV$, annihilation σ can be considerably enhanced via Sommerfeld.

- If the force-carriers are non-Abelian gauge bosons, one has excited states for explaining anomalies from INTEGRAL (via xDM) and DAMA (via iDM).

2.2 Sommerfeld Enhancement and DM interacting via Yukawa

It is possible to have *Sommerfeld Enhancement* (see appendix B) when the particle has an **attractive force carrier** with a **Compton wavelength longer than** $(\alpha M_{DM})^{-1}$, which means that there are **dark matter bound states**.

From quantum mechanics, for a s-wave annihilation in non-relativistic limit, the reduced two-body wavefunction obeys the radial Schroedinger equation (v is the velocity of each particle in the center-of-mass):

$$\frac{1}{m_\chi}\psi''(r) - V(r)\psi(r) = -m_\chi v^2 \phi(r). \tag{2.1}$$

For scalar ϕ the potential is the Yukawa potential:

$$V(r) = -\frac{\lambda^2}{4\pi r} e^{-m_\phi r} \tag{2.2}$$

The Sommerfeld enhancement in the scattering cross-section due to the potential is:

$$S = \left(\frac{\frac{d\phi}{dr}(0)}{k}\right)^2 = |\psi(\infty)/\psi(0)|^2. \tag{2.3}$$

Rescaling (2.1):

$$\alpha = \frac{\lambda^2}{4\pi}, \tag{2.4}$$

$$\epsilon_\mu = \frac{v}{\alpha}, \tag{2.5}$$

$$\epsilon_\phi = \frac{m_\phi}{\alpha m_\chi}, \tag{2.6}$$

$$r' = \alpha m_\chi r, \tag{2.7}$$

One can write it as:

$$\phi''(r') + \left(e_v^2 + \frac{1}{r'} e^{-\epsilon_\phi r'} \right) \psi(r') = 0. \tag{2.8}$$

In the limit $\epsilon_\phi \to 0$, the effective potential is just Coulomb, and (2.8) gives the enhancement factor:

$$S = |\psi(\infty)/\psi(0)|^2 = \frac{\pi/\epsilon_v}{1 - e^{-\pi/\epsilon_v}} \tag{2.9}$$

- For nonzero ϵ_ϕ, the Sommerfeld enhancement saturates at low velocity.
- For the interesting range $m_\phi \sim 100$ MeV-GeV, $\epsilon_\phi \sim 10^{-2} - 10^{-1}$, and the Sommerfeld enhancement is $\sim 10^3 - 10^4$.

2.3 Models of the Sommerfeld Force

There are three candidates for the possible light force-carrier, with massive degree of freedom that couples to DM:

- **Light scalar field**, with a attractive interaction. However, it is unnatural for the scalar to stay light unless the DM sector is very symmetric.
- **A pseudoscalar** π **with a goldstone-like** coupling to the matter $1/F J_\mu \partial^\mu \pi$, the scalar can be light.
- **Spin-1gauge fields**, arising from some dark gauge symmetry G_{dark}. The break scale would be $G_{dark} \sim$ GeV.

2.4 DM coupling to a Scalar Field

The process for which an attractive interaction by a scalar field produces Sommerfeld enhancement can be summarized:

1. The symmetry breaking dominates the asymptotic states, and dark matter must be part of a *multiplet with at least two states*, since spin-1 particles cannot have coupling to a single neutral state.
2. The gauge symmetry breaking leads to a *mass splitting between the states*, which dominates the long-distance behavior of the theory. It determines which state is the lightest and initial state for collisions leading to annihilations.
3. If the mass spliting between states is small enough compared to the kinetic energy of collision, the gauge-partner of DM states will be active in the collisions at distances smallers than the gauge bosons masses, and the gauge breaking is negligenciable.

4. The asymptotic state are the linear combinations of positive and negative charged gauge eigenstates: the two-body wavefunction is a linear combination of attractive and repulsive channels. There will be still a Sommerfeld enhancement for ordinary WIMP annihilation.

5. To *not spoil nucleosynthesis*, new states cannot be very light (mass less than < 10 MeV), and all light states in the dark sector has mass $\sim$ GeV. This can be not stable, so lightest can only decay back to SM, producing high energy e^+e^-.

6. A scalar ϕ can couple with a dillaton-like coupling $\phi F^{\mu\nu}F_{\mu\nu}$, producing photons and hadrons (via gluons) and failing to produce hard e^+e^- spectrum. However, if there is mixing with Higgs, ϕ acquires a vev $\langle\phi\rangle \sim m_\phi$, and ϕ will decay into the heaviest fermion pair available.

7. For $m_\phi < 200$ MeV, it will decay directly to e^+e^- while for 200 Mev$<$ $m_\phi < 250$ MeV, ϕ will decay dominantly to $\mu^+\mu^-$. Above that, pion modes will dominate. Both give good fit to PAMELA, but the former fits better PAMELA+ATIC.

2.5 DM coupling to a Pseudoscalar Field

A pseudoscalar which does not yield a Sommerfeld enhancement could be in the new sector, and this particle would couple to the heaviest particle available, or through the axion. The decays would be similar to the scalar.

2.6 DM coupling to a Vector Boson

A vector boson mixes with eletromagnetism via $F_{\mu\nu}F^{\mu\nu}$. This operator will cause a vector ϕ_μ to couple directly to the charge:

- For $m_\phi < 2m_\mu$ it will decay to e^+e^-.
- For $2m_\mu < m_\phi < 2m_\pi$ it will decay equally to e^+e^- and $\mu^+\mu^-$.
- Above $2m_\pi$ it will decay also in $\pi^+\pi^-$, with no direct decay to π^0.

2.7 A Non-Abelian G_{DARK}

A 800 GeV WIMP interacting via a particle that couples to charge is constrained to the fact that the ϕ boson is light and couples to the electomagnetic vector current. The cross-section per nucleon is (from [1]):

$$\sigma_0 = \frac{16\pi Z^2 \alpha_{SM}\alpha_{DARK}\epsilon^2\mu_{ne}^2}{A^2 m_\phi^4}, \tag{2.10}$$

where α_{DARK} is the coupling of ϕ to the dark matter, ϵ is the kinetic mixing, μ_{ne} is the reduced mass of the DM-nucleon system, and α_{SM} is the standard model electromagnetic coupling constant. With this parameters, this σ_0 is excluded by the present CDMS.

2.7.1 XDM from a Non-Abelian Symmetry

The XDM excitations could occur in the center of the galaxy by inelastic scattering $\chi\chi \to \chi\chi^*$. If $\delta = m_{\chi^*} - m_\chi > 2m_e$ the decay $\chi^* \to \chi e^+ e^-$ can produce the excess of 511 KeV x-rays seen from the galactic center by INTEGRAL. However, It is necessary a large σ to produce the large number of positrons, together with a boson with mass of the order of the momentum transfered, i.e., $m_\phi < M_\chi v \sim$ GeV. This is the same scale required for the Sommerfeld enhancement.

One can assume DM transforming under a *non-abelian gauge symmetry* to have excited states. Although an excited state can be present with U(1), this does not explain the INTEGRAL signal. Assuming DM is *Majorana fermion*, it must transform as a *real* representation of the gauge symmetry.

For a non-Abelian symmetry, the smallest representation will be three-dimensional, such as a triplet of SU(2), allowing the scattering $\chi_1\chi_1 \to \chi_2\chi_3$.

If m_3 is split from $m_2 \sim m_1$ by an amount $\delta \sim$ MeV, we have XDM for INTEGRAL sinal. Because the gauge symmetry is from Higgs, there are a splitting between different states in the DM multiplet. The gauge breaking in the gauge boson masses would lead, one loop-level, to splittings between different dark matter states, which is the the MeV splittings needed for XDM. To have χ_1 and χ_2 similar in mass, which can occur if the breaking pattern preserves symmetry, it gives $\delta \sim$ 100 - 200 KeV which is the range for inelastic dark matter of DAMA.

Finally, the lagrangian is on the form:

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{Dark} + \mathcal{L}_{Mix}. \tag{2.11}$$

One makes $G_{Dark} = SU(2) \times U(1)$, with gauge bosons $\omega_{\mu I}$ and b_μ, collectively referred as $a_{\mu i}$, and the DM multiplet χ transforming as a triplet under $SU(2)$ and neutral under $U(1)$, being either a scalar or a fermion.

With Higgs breaking symmetry, working in unitary gauge, the tree-level dark sector Lagrangian:

$$\mathcal{L}_{Dark} = \mathcal{L}_{GaugeKin} + \frac{1}{-} m_{ij}^2 a_i^\mu a_{\mu j} + ... \tag{2.12}$$

where the m_{ij}^2 makes massive the dark spin-1 fields. At one loop, this broken gauge symmetry will split between the 3 real DM states. The interaction between the two sectors is via kinetic mixing between the new U(1) and the photon (mixing with hypercharge):

$$\mathcal{L}_{Mix} = \frac{1}{2}\epsilon b_{\mu\nu} F^{\mu\nu}. \tag{2.13}$$

At PAMELA and ATIC, the non-abelian self-couplings of the vector bosons can have an effect on the annihilation process. For large α_{Dark}, the gauge bosons lead to a shower.

If DM is charged under $SU(2) \times U(1)$, it can be a subdominat component of annihilations into W^+W^-, larger than the s-wave limited thermal cross-section, yielding photon signals from hadronic shower.

Table 1: Spectrum of XDM

$\chi_1[m_\xi]$
$\chi_2[m_\xi + (\alpha m_{\phi'} \sim 100 \text{ KeV})]$
$\chi_3[m_\xi + (\alpha m_\phi \sim 1 \text{ MeV})]$

If the excess of e^+e^- at PAMELA and ATIC are from DM, it is not sufficient a modification of SM (such as a neutralino) in the *minimal supersymmetric standard model* (MSSM), but it is needed leptonic annihilation modes with large σ. It can be a new light state, decaying dominantly to leptons and could be produced at LHC, leading to boosted pairs of leptons as a generic signature.

3 Missing E_T signatures and Dark Matter Connection

LHC might find signatures for low mass dark photons, such as mixing with visible photon, and decaying to $\mu^+\mu^-$. One example of such event producing dark matter is illustrated at figure 2.

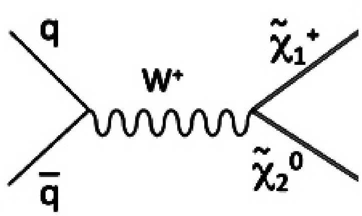

Figure 2: An example of event for large missing energy due to darkinos (X) at the Tevatron (this is the first part of a diagram).

This Feynman diagram would be given by

$$u^q(p)(-g\gamma^\mu t^a)\bar{u}^q(p)(\frac{-ig_{\mu\nu}}{k^2 - M_W^2})u^{\tilde{X}_1^+}(p)(-g\gamma^\nu)t^a\bar{u}^{\tilde{X}_2^0}(p).$$

References

[1] N.Arkani-Hamed et al., A Theory of Dark Matter, http://arxiv.org/abs/0810.0713.

[2] H. Baer, TASI 2008 Lectures: Collider Signals, http://arxiv.org/abs/0901.4732v1.

[3] K. Zurek, TASI 2009 Lectures: Searching for Unexpected Physics at the LHC, http://arxiv.org/abs/1001.2563v1.

[4] Wikipedia http://wikipedia.org.

[5] Quantum Field Theory, Peskin.

[6] Quantum Field Theory, Sterman.

[7] Private talk with Prof. Patrick Meade.

A Data from Astrophysics Experiments

Thermal WIMPs are one of the most attractive candidates for Dark Matter (DM), they appear in theories of weak-scale beyond the standard model, and they give the appropriate *relic abundance*. If dark matter annihilates to some set of standard model states, cosmic ray detectors such as PAMELA, ATIC and FGST can detect it. Recent experiments have been appointed to an excess of positron and electrons at GeV that cannot be explained by supernova shocking and interaction of cosmic rays:

Table 2: Astrophysics Experiments in Cosmic Rays and Dark Matter.

Project	**Possible correlation to DM/ DM Annihilation**
PAMELA	Excess of positron from 10-50 GeV.
ATIC	Excess in $e^{\pm}$ at 500-800 GeV.
WMAP	Hard component (haze) not spatially correlated with any mechanism.
EGRET	Excess of 10-50 GeV at the galaxy center.
INTEGRAL	Excess of 511 KeV electrons at the galaxy center, **XDM**.
DAMA	Signal still compatible with null results, **iDM**.

Besides possible other explanations, the fact that there are multiple sources pointing for a high $e^{\pm}$ emissions may be a connection to DM. However, in perturbative annihilation, s-waves dominate in the late universe, and for a weak-scale thermal particle, the relic abundance in a s-wave annihilation has a *very low cross section*, which is a upper limit of signal that could be observed. Therefore, a model for DM requires a weak-scale DM that annihilates to SM with a sizable σ (DM cannot be a gravitino in low-energy SUSY). This larger annihilation cross-section can be explained by DM coupled to new light states ($\sim$ 1 GeV), with a *Sommerfeld enhancement*.

The DM arises then from a multiplet of vector-like states, with some/all flavors symmetry gauged, which is a natural extension of SM with low-energy SUSY. Finally, any model of dark matter will then have the challenges of having:

- A large σ (larger than the allowed by thermal relic abundance).

- A large σ into leptons (DM would annihilate directly to leptons).
- A low σ into hadrons.

A.1 Astrophysics Experiments

PAMELA: The **Payload for Antimatter Matter Exploration and Light-nuclei Astrophysics** is a satellite-based experiment dedicated to the detection of cosmic rays, attached on Earth orbiting satellite, and its particular focus is the antimatter component (positrons and antiprotons). It might detect evidence of dark matter annihilation. Preliminary data shows an excess of positrons in the range 10-50 GeV. No excess of antiprotons was found. This is inconsistent with predictions from most models of dark matter sources, in which the positron and antiproton excesses are correlated.

ATIC: The **Advanced Thin Ionization Calorimeter** is a balloon-borne instrument flying over Antarctica to measure the energy/ composition of cosmic rays.

Preliminary data found a extra number of high energy electrons in the range 500-800 GeV; these electrons were in excess of those expected from the galactic background. The electrons could originate from a nearby pulsar or other astrophysical object, or the from collisions of Dark Matter particles, WIMP particles of mass $\sim$ 620 GeV.

EGRET: The **Energetic Gamma Ray Experiment Telescope** was one of four instruments on NASAs Compton Gamma Ray Observatory satellite. Since lower energy gamma rays cannot be accurately detected on Earths surface, EGRET was built to detect gamma rays while in space. EGRET was created for the purpose of detecting and collecting data on gamma rays ranging in energy level from 30 MeV to 30 GeV.

INTEGRAL: The **European Space Agency's INTErnational Gamma-Ray Astrophysics Laboratory** is an operational Earth satellite, launched in 2002 for detecting some of the most energetic radiation coming from space. It is the most sensitive gamma ray observatory ever launched.

DAMA/LIBRA: The **DAMA/LIBRA** experiment is constructed to detect dark matter using the direct detection. The setup consists of scintillation thallium-doped sodium iodide (NaI) radioactivity pure crystals. Nuclei recoiling after a collision emits photons that are detected. A detected nuclear recoil can be caused by dark matter particles or by the background (thermal neutrons, radioactivity or cosmic radiation). The revolution of the Earth around the Sun can cause an annual modulation of the dark matter flux. This should give rise to an annual modulation in the detected recoils and thus provides a simple way to extract a dark matter signal from the background.

FGST: The **Fermi Gamma-ray Space Telescope** (formerly GLAST) is a space observatory being used to observe gamma-ray, studying astrophysical and cosmological phenomena such as active galactic nuclei, pulsars, other high-energy sources and dark matter (e.g. by looking for an excess of gamma rays from the center of the Milky Way and early Universe.)

WMAP: The **Wilkinson Microwave Anisotropy Probe** is a spacecraft which measures differences in the temperature of the Cosmic Microwave Background Radiation, In this Λ-CDM model of the universe, the age of the universe is 13.75 ± 0.11 billion years. The content of the universe presently consists of $4.56\% \pm 0.15\%$ ordinary baryonic matter; $22.8\% \pm 1.3\%$ Cold dark matter (CDM) that neither emits nor absorbs light; and $72.6\% \pm 1.5\%$ of dark energy in the form of a cosmological constant that accelerates the expansion of the universe. The contents point to a flat Euclidean flat geometry, with the ratio of the energy density in curvature to the critical density $0.0179 < \Omega_k < 0.0081$.

A.2 The Relic Abundance

Weak scale dark matter gives naturally the relic abundance (*WIMP miracle*). The DM density freezes out at temperature T_{f_o} when the annihilation cross-section becomes $\sim$ Hubble expansion. The DM energy density at freeze-out is:

$$\rho_{DM} \sim T_{f_o}^4 \left(\frac{m_{DM}}{T_{f_o}} \right)^{5/2} e^{-m_{DM}/T_{f_0}}. \tag{A.1}$$

One can compare to the observed DM to photon energy density ratio:

$$\eta_\gamma = \frac{\rho_{DM}^0}{T_\gamma^4} = \frac{T_{f_o}}{T_\gamma} \left(\frac{m_{DM}}{T_{f_o}} \right)^{5/2} e^{-m_{DM}/T_{f_0}}, \tag{A.2}$$

where ρ_{DM}^0 is the observed DM density and T_γ the CMB photon energy, both at present time.

For $M_{DM}/T_{f_o} \approx 1/25$, one obtains the observed relic density ($\sigma v \sim 3 \times 10^{-26} cm^3/s \sim \frac{1}{TeV^2}$), which satisfies the typically annihilation cross-sections scale:

$$\sigma_{ann} v \sim \frac{g^4}{m_{DM}^2}, \tag{A.3}$$

where g is the coupling of the DM to the mediator of annihilation.

In abelian hidden sectors weakly coupled to the Standard Model (section1) through light gauge mediator, one sees from A.3 that the relic abundance is also obtained to MeV dark matter, even if it is not a weak scale DM. These hidden sector models are relevant for PAMELA and FGST, because light dark forces can give rise to boosted annihilation σ when DM becomes non-relativistic.

B Sommerfeld Enhancement

The Sommerfeld enhancement is an effect in nonrelativistic quantum mechanics. Considering a non relativistic particle moving around some origin. There is an interaction Hamiltonian $H_{ann} = U_{ann}\delta^3 r$ localized on the origin. Imagining the particle moving in the z direction so the wavefunction is $\psi_k^{(0)}(x) = e^{ikz}$, the rate for this process is proportional to $|\psi_k^{(0)}(0)|^2$.

Now supposing one has a central potential $V(r)$ attracting or repelling the particle to the origin. One could treat V perturbatively, but at small velocities the potential may not be a small perturbation and can distort the wavefunction. This can be determined by the Schroedinger equation:

$$-\frac{1}{2M}\nabla^2\psi_k + V(r)\psi_k = \frac{k^2}{2M}\psi_k,$$

with boundary condition enforcing that the perturbation can only produce outgoing spherical waves as $r \to \infty$:

$$\psi \to e^{ikz} + f(\theta)\frac{e^{ikr}}{r}.$$

Since the annihilation is taking place locally near r=0, the only effect of the perturbation V is to change the value of the modulus of the wave-function at the origin relative to its unperturbed value:

$$\sigma = \sigma_0 S_k,$$

where the Sommerfeld enhacement factor is:

$$S_k = \frac{|\psi_k(0)|^2}{|\psi_k^{(0)}(0)|} = |\psi_k(0)|^2.$$

Any solution of the Schoedinger equation with rotational invariance around z can be expanded as:

$$\psi_k = \sum_l A_l P_l(\cos\,\theta) R_{kl}(r),$$

where $R_{kl}(r)$ are the continuum radial functions associated with angular momentum l. Using the asymptotic expansion of e^{ikz}, one can determines the expansion :

$$\psi_k = \frac{1}{k}\sum_l i^l(2l+1)e^{i\delta_l}P_l(\cos\theta)R_{kl}(r).$$

Thus, one has:

$$S_k = \left|\frac{R_{k,l=0}(0)}{k}\right|^2 = \left|\frac{\frac{d\chi_k}{dr}(0)}{k}\right|^2$$

B.1 Attractive Coulomb Potential

$$V(r) = -\frac{\alpha}{2r}.$$

Rescalling into natural variables:

$$r = \alpha^{-1}M^{-1}x,$$

the Schroedinger equation becomes

$$-\chi'' - \frac{1}{x}\chi = \epsilon^2\chi,$$

where we define $\epsilon_v = v/a$. The Sommerfield Enhancement is then:

$$S_k = \left| \frac{\frac{\pi}{\epsilon_v}}{1 - e^{-\frac{\pi}{\epsilon_v}}} \right|.$$

B.2 Cosmological Implications

B.2.1 Early Universe

- The particle leaves thermal equilibrium long before the Sommerfeld enhancement turn on, when the expansion parameter $\alpha/v = 1/\epsilon_v$ is large.
- Dark matter typically decouples at $T_{CMB} \sim m_\chi/20$ or $v \sim 0.3c$, when the enhancement has not turned on yet.
- One has then the perturbative annihilation cross-section $\sigma \sim \alpha^2/m_\chi^2$ giving the usual successful thermal relic abundance.

B.2.2 Later time

- The Sommerfeld enhancement turns on and the annihilations begin to scale as $a^{-5/2}$ (before kinetic decoupling) or a^{-2} (after decoupling).
- From decoupling until matter-radiation equality, dark matter annihilation produces a uniform amount of energy per comoving volume per Hubble time and one effect for this could be the polarization of the CMB.