Scattering, Diffraction and Radiatiom

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Contents

1 Scattering

An incident radiation field can be seen as inducing electric and magnetic poles that oscillate in definitive phase relationship with the incident wave, and radiate energy in directions others than the direction of incidence (coherent superposition).

$$E_{inc} = \epsilon_0 E_0 e^{ikn_0.x},$$
$$H_{inc} = n_0 \times E_{inc}/Z_0,$$

where n_0 is the incident direction, ϵ_0 is the polarization vector and $k = \omega/c$.

These fields induce dipole moments **p** and **m** in the small scatterer. These dipoles radiate to all directions.

Far away from the scatterer, the fields are:

$$E_{sc} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} \left((n \times p) \times n - n \times m/c \right),$$
$$H_{sc} = n \times E_{sc}/Z_0.$$

The power radiated is the *differential scattering cross section*:

$$\frac{d\sigma}{d\Omega} = \frac{r^2 \frac{1}{2Z_0} |e^*.E_{sc}|^2}{\frac{1}{2Z_0} |\epsilon_0^* E_{inc}|^2}.$$

1.1 Scattering by a Small dielectric Sphere

Electric Dipole	$p = 4\pi\epsilon_0 (\frac{\epsilon_r - 1}{\epsilon_r + 2}) a^3 E_{inc}$
Magnetic Dipole	0
Cross Section	$\frac{d\sigma}{d\Omega} = k^4 a^6 \vert \frac{\epsilon_r - 1}{\epsilon_r + 2} \vert^2 \vert \epsilon^*.\epsilon_0 \vert^2$ $\frac{d\sigma_\parallel}{d\Omega} = \frac{k^4 a^6}{2} \vert \frac{\epsilon_r - 1}{\epsilon_r + 2} \vert^2 cos^2\theta$ $\frac{d\sigma_\perp}{d\Omega} = \frac{k^4 a^6}{2} \vert \frac{\epsilon_r - 1}{\epsilon_r + 2} \vert^2$
Polarization	$\Pi(\theta) = \frac{sin^2\theta}{1 + cos^2\theta}$
Total Scattering Cross Section	$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \vert \frac{e_r - 1}{e_r + 2} \vert^2$

The scattered radiation is linearly polarized in the plane defined by the dipole moment direction ϵ_0. The blue sky corresponds to the maximum polarization of $\Pi(\theta)$, at the angle $\theta = \frac{\pi}{2}$.

1.2 Scattering by a Small conducting Sphere

There is a strong backward peaking by interference between dipole. The polarization is maximum, equals to 1, when $\theta = 60^o$.

Electric Dipole	$p = 4\pi\epsilon_0 a^3 E_{inc}$
Magnetic Dipole	$m = -2\pi a^3 H_{inc}$
Cross Section	$\frac{d\sigma}{d\Omega} = k^4 a^6 \lvert\epsilon^*.\epsilon_0 - \frac{1}{2}(n \times \epsilon^*).(n \times \epsilon_0)\rvert^2$
	$\frac{d\sigma_\parallel}{d\Omega} = \frac{k^4 a^6}{2}\lvert cos\theta - \frac{1}{2}\rvert^2$
	$\frac{d\sigma_\perp}{d\Omega} = \frac{k^4 a^6}{2}\lvert 1 - \frac{1}{2}cos\theta\rvert^2$
Polarization	$\Pi(\theta) = \frac{3sin^2\theta}{5(1+cos^2\theta)-cos\theta}$

1.3 The General Theory

The medium through an electromagnetic wave is passing has spatial variations in the electromagnetic proprieties ($D \neq \epsilon_0 E$ and $B \neq \mu_0 H$ in some regions), thus the wave is scattering.

From Maxwell equations, the wave equation is:

$$\nabla^2 D - \mu_0\epsilon_0 \frac{\partial^2 D}{\partial t^2} = -\nabla \times \nabla(D - \epsilon_0 E) + \epsilon_0 \frac{\partial}{\partial t}\nabla \times (B - \mu_0 H).$$

From the time dependence $e^{-i\omega t}$:

$$(\nabla^2 + k^2)D = -\nabla \times \nabla \times (D - \epsilon_0 E) - i\epsilon_0\omega\nabla \times (B - \mu_0 H).$$

A formal solution is :

$$D = D^0 + \frac{1}{4\pi}\int d^3x' \frac{e^{ik|x-x'|}}{|x-x'|}(\nabla' \times \nabla' \times (D - \epsilon_0 E)),$$
$$D = D^0 + \frac{1}{4\pi}\int d^3x' \frac{e^{ik|x-x'|}}{|x-x'|}(i\epsilon_0\omega\nabla' \times (B - \mu_0 H)),$$

The field away from the scattering region is:

$$D \to D^0 + \frac{A_{sc}e^{ikr}}{r},$$
$$A_{sc} = \frac{k^2}{4\pi}\int d^3x e^{-iknx}(n \times (D - \epsilon_0 E) \times n)),$$
$$A_{sc} = \frac{k^2}{4\pi}\int d^3x e^{-iknx}(-\frac{\epsilon_0\omega}{k} n \times (B - \mu_0 H)).$$

The cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{|E^* A_{sc}|^2}{|D^0|^2}.$$

1.4 Blue Sky

1. The magnetic dipoles are approximate equal to zero.
2. The electric dipole of the molecules are $p_j = \epsilon_0 \gamma_{mol} E(x_j)$.
3. The effective variation on ϵ is $\delta\epsilon(x) = \epsilon_0 \sum_j \gamma_{mol} \delta(x - x_j)$.
4. The cross section is $\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} |\gamma_{mol}|^2 |\epsilon^* . \epsilon_0|^2 \mathcal{F}(q)$.
5. The *Raylengh Scattering*, which is the scattering section per molecule: $\sigma \sim \frac{k^4}{6\pi N^2} |\epsilon_r - 1|^2 \sim \frac{2k^4}{3\pi N^2} |n - 1|^2$, for $|n - 1| < 1$.
6. In a thickness dx of gas, the loss of the incident flux is $N\sigma dx$. The incident beam has $I(x) = I_0 e^{-\alpha x}$, where α is the absorption/attenuation coefficient $\sigma = N\sigma \sim \frac{2k^2}{3\pi N} |n - 1|^2$.

1.5 Thompson Scattering

If a plane wave of monochromatic electromagnetic radiation is incident on a free particle of charge e and mass m, the particle will accelerated and so emit radiation. This radiation will be emitted in directions other than of the incident plane wave, but for nonrelativistic motion of the particle it will have the same frequency as the incident radiation.

$$E \sim E_0 \hat{x} e^{-i\omega t + ikz}, B \sim E_0 \hat{y} e^{-i\omega t + ikz},$$

If one has $\frac{v}{c} \ll 1$, B does not contribute and from Drude's Model, doing $a = -\omega^2 x$:

$$a(t) = -\omega^2 \text{ Re } (-\frac{eE_0}{m} \frac{e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\gamma\omega}),$$
$$\langle a^2 \rangle \sim \frac{1}{T} \int dt a^2(t) = \frac{1}{2} \frac{1}{|\Delta|^2}.$$

One then calculates (where P is the power loss):

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{4\pi} \frac{e^2}{c^3} \sin^2\theta \frac{c^2 E_0^2}{2m^2} \frac{\omega^4}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}, \\
&= \frac{c}{8\pi} (\frac{e^2}{mc^2})^2 E_0^2 \frac{\omega^4}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2} \sin^2\theta, \\
&= \gamma^2 e F(\omega) sin^2\theta \frac{E_0^2}{8\pi}.
\end{aligned}$$

The energy flux is $\frac{cE_0^2}{8\pi}$. The Pointying vector is:

$$S = \frac{c}{4\pi} E \times B = \frac{c}{4\pi} E^2 z,$$
$$\langle |S|^2 \rangle_{time} = \frac{c}{8\pi} E_0^2.$$

With the scattering wave vector $k = \frac{\omega r}{c}$, the scattering amplitude F can be found by:

$$\frac{dP/d\Omega}{\langle |P| \rangle} = \frac{d\sigma}{d\Omega} = re^2 \sin^2\theta \frac{\omega^4}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2},$$
$$= re^2 F(\omega) \sin^2\theta.$$

The total cross section (from optical theorem, called the Thomson cross section) is then:

$$\sigma_{total} = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{4\pi}{k} \text{ Im } F(\omega, \theta = \frac{\pi}{2}),$$
$$= \frac{8\pi}{3} (\frac{e^2}{mc^2})^2.$$

1.5.1 The Classical Cross Section

$$\frac{\textbf{ENERGY RADIATED IN } \Omega}{\textbf{TOTAL ENERGY FLUX}} = \frac{d\sigma}{d\Omega},$$

$$\frac{d\sigma}{d\Omega} = re^2 \sin^2\theta F(\omega).$$

Where $F(\omega)$ is the force factor:

$$F(\omega) = \frac{\omega^4}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}.$$

The first term characterizes radiation and the second, matter. One has then two case:

- $\omega \gg \omega_0$: $F(\omega) \sim 1$, $\frac{d\sigma}{d\Omega} \sim re^2 \sin^2\theta$.
- $\omega \ll \omega_0$: $F(\omega) \sim \frac{\omega^4}{\omega_0^4}$, Raylengh.

  Obs: Blue scattered 16 times more then red:

$$\frac{\sigma_{red}}{\sigma_{blue}} \sim (\frac{\lambda_{blue}^4}{\lambda_{red}}) = 2^{-4}$$

2 Diffraction

2.1 The Diffraction in a Hole

The interaction of the light to the obstacle means classical radiation (dipole-like emission). Supposing θ the angle to the plane of the hole, from the dipole formula:

$$\frac{E_T}{E_R} = \frac{V_T \cos\,\theta}{c\tau},$$
$$E_T = \frac{e}{c^2 R}\sin\,\theta a.$$

From the Drude's model, one has:

$$a(t) = -\omega^2 x(t) = \frac{e}{m}\frac{\omega^2}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0^{-i\omega t},$$
$$x(t) = -\frac{e}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0^{-i\omega t}.$$

The scattering field:

$$E_s(t) = \frac{c^2}{me^2}\frac{\omega^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\frac{\sin\,\theta}{r}E_0 e^{-i\omega t + \frac{i\omega}{c}r},$$

where the last exponential is the retardation. To determine the interaction between in-wave and the obstacle, one uses the *principle of complementarity*, thinking about plugging the hole:

$$E_{screen} + E_{plug} = 0$$

$$dE_s(t,r) = \frac{\theta e^2}{mc^2}\frac{\omega^2 sin\theta}{\omega_0^2 - \omega - i\gamma\omega}E_0\frac{e^{-i\omega t - \omega r/c}}{r}\mu dx'dy'.$$

The last part is the density of the plug. Redefining the micro-information by:

$$dE_s(t,r) = CE_0\,\sin\,\theta\frac{e^{-i\omega t + \frac{i\omega r}{c}}}{r}dx'dy',$$

one can integrate and gets for $z \gg \lambda$ or $kz \gg 1$:

$$C = -\frac{ik}{2\pi}.$$

The diffraction is then given by the Kirchhoff's Integral:

$$E_{screen}(t,r) = (-\frac{ik}{2\pi})E_0 \oint_{hole} dx'dy' \sin\theta e^{-it\omega+ikr}.$$

If $z \gg \lambda$ and $z \gg \sqrt{Hole}$, $\theta \approx \frac{\pi}{2}$:

$$E_{screen}(k,r) \sim -\frac{ik}{2\pi}E_0 \sin\theta e^{-i\omega t} \oint_{hole} dx'dy' \frac{e^{ikr}}{r}. \tag{2.1}$$

2.2 The Fraunhofen Diffraction

From (2.1), now one writes r on spherical coordinates for the diffraction in a hole:

$$r = \sqrt{(x-x')^2 + (y-y')^2 + z^2} = \sqrt{r_0^2 - 2xx' - 2yy' + x^2 + y^2}.$$

By Taylor expansion ($a \ll z$):

$$r = r_0 - \frac{xx'}{r_0} - \frac{yy'}{r_0} + \ \dots \ ,$$
$$\text{where } r_0 = \sqrt{x^2 + y^2 + z^2}.$$

In this case, the scattering electric field is:

$$E_s(t,x) = (-\frac{ik}{2\pi})E_0 \frac{e^{-i\omega t + ikr_0}}{r_0} \int dx'dy' e^{-\frac{ik}{r_0}xx' - \frac{ik}{r_0}yy'}.$$

By substituting variables, one finds the dependence on *Bessel functions* for the diffraction field in a disk:

$$E_s(t,x) \sim (-ik)E_0 \frac{e^{-i\omega t + ikr_0}}{r_0} \frac{J_1(u)}{u}.$$

It turns out that this is the modulation of the hole, and the minima will be the zero of the Bessel functions. This will give the spectrum of the diffraction in the hole (figure 1).

The transmitted energy by the Poynting vector is:

$$\begin{aligned} S &= \frac{c}{4\pi} = \frac{c}{4\pi}E^2, \\ \bar{S} &= \frac{c}{8\pi}E^2 = \frac{cE^2}{8\pi}\Big(\frac{k^2a^4}{r_0^2}\Big)^2\Big(\frac{J_1^2(u)}{u}\Big)^2 \end{aligned}$$

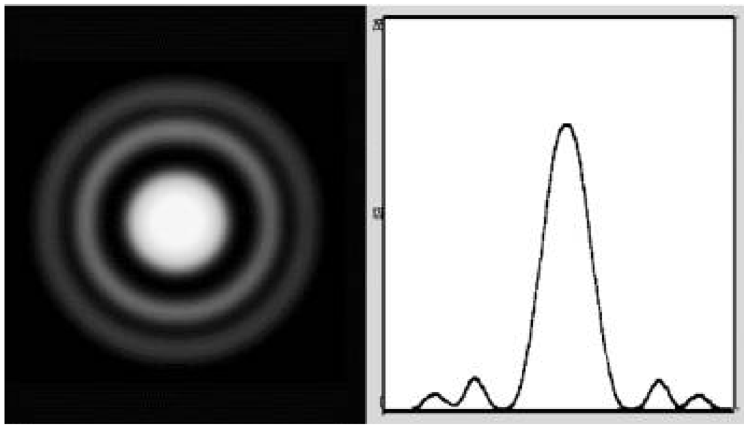

Figure 1: Fraunhofen Diffraction and the Bessel spectrum.

2.3 Fresnel Diffraction

A closer look at the Fraunhofen center revels more intrinsic structure. In the center, the phase is zero, and the field is like a plane wave. At the center one has

$$\begin{aligned} r &= \sqrt{(x-x')^2+(y-y')^2+z^2}, \\ &\sim r_0 + \frac{x'^2+y'^2}{2r+0} + \mathcal{O}(a^3), \end{aligned}$$

where the second term is the dominant phase, and it is like a plane wave. The diffraction formula is:

$$\begin{aligned} E_s(t,z) &= -\frac{ik}{2\pi} E_0 \frac{e^{-i\omega t+ikr_0}}{r_0} \int_a^a \rho' dp' \int_0^{2pi} d\phi' e^{\frac{ik}{2r_0}\rho'} \\ &= -\frac{ik}{2} E_0 \frac{e^{-i\omega t+ikr_0}}{r_0} \int_o^a \partial\rho'^2 e^{\frac{ik}{2r_0}\rho'^2} \\ E_s(t,z) &= E_{im}(t,z)(1-e^{i\frac{ka^2}{2r_0}}). \end{aligned}$$

The mean radiation is:

$$\bar{S} = \frac{cE_0^2}{2\pi} sin^2(\frac{ka^2}{4r_0}).$$

In the spectrum, with $\frac{ka^2}{4r_0} = n\pi \ \frac{1}{n} = \frac{4r_0\pi}{kna^2}$, and for every integer number there is a spot.

3 Radiation

3.1 Retarded Potential

Recalling the fully retarded solution using the Cauchy data set, the retarded potential is:

$$A_\mu(t,\bar{x}) = \theta(t).\frac{1}{c}\int dx' \frac{J_r(t',x')}{|x-x'|},$$
$$t' = t - \frac{|x-x'|}{c} = t - \frac{R}{c},$$
$$A_0(t,x) = \phi(t,x) = \int dx \frac{\rho(t',x')}{|x-x'|}.$$

The *Liennard-Viechert* Potential is:

$$A_\mu(t,x) = \frac{q}{1-\frac{\hat{n}v'}{c}} \frac{1}{|x-x'|} (\frac{v'}{c})^0_j.$$

3.2 Radiation Field

$$E(x,t) = -\nabla\phi(t,x) - \frac{\partial A}{c\partial t}(t,x),$$
$$B(t,x) = \nabla \times A(t,x).$$

Since it is just a Coulomb field, $\frac{1}{r^2}$ are ignored. The potential for the particle in motion is:

$$\phi(t,x) = \frac{q}{R}\frac{1}{1-\frac{n'v'}{2}},$$
$$A(t,x) = \frac{v'}{c}\phi(t,x)$$

The derivative is:

$$\begin{aligned}
\frac{\partial t'}{\partial t} &= \partial_t\Big(t - \frac{|x-x'(t)|}{c}\Big) \\
&= 1 - \frac{1}{c}\partial_t \frac{|x'-x|}{\sqrt{(x'-x)^2}} \\
&= \frac{1}{1-\frac{\hat{n}v}{c}}.
\end{aligned}$$

$$\begin{aligned}
\nabla t' &= \nabla\Big(t - \frac{|x-x'|}{c}\Big) \\
&= -\frac{\hat{n}'}{c} + \frac{\hat{n}'v'}{c}\nabla t'.
\end{aligned}$$

$$
\begin{aligned}
\nabla t' &= \frac{-\frac{\hat{n}'}{c}}{1-\frac{\hat{n}'v'}{c}},\\
d_t t' &= \frac{1}{1-\frac{\hat{n}'v'}{c}},\\
E(t,x) &= -\nabla\phi - \frac{1}{c}d_t H,\\
A(t,x) &= \frac{q\frac{v'}{c}}{1-\frac{\hat{n}'v'}{c}|x-x'|}.
\end{aligned}
$$

One only takes the $\frac{1}{R}$ part, i.e, the radiation field that survives $R \to \infty$ (far field). Thus, one has:

$$
\begin{aligned}
\nabla\phi &\approx \frac{q}{R}\frac{-1}{(1-\frac{m'v'}{2})^2}\Big[\nabla(-\frac{m'v'}{c}) = [\partial_t(\frac{-nv}{c})]\nabla t'\Big],\\
&\approx \frac{\frac{\hat{n}'a'}{c}}{(1-\frac{n'v'}{c})}\left(\frac{-\hat{n}/c}{R}\right).
\end{aligned}
$$

It results on the electric field:

$$
E \approx \frac{1}{Rc^2}\frac{1}{(1-\frac{v'u'}{c})^3}\hat{n}\times((\hat{n}-\frac{v'}{c})\times a').
$$

The magnetic field can be found by $B = \nabla\times A$, it is perpendicular to E and has the same magnitude$|B| = |E|$:

$$
B \approx \frac{q}{R}a.
$$

3.3 The Larmor Formula (non-relativistic $\frac{v}{c} \ll 1$)

For an accelerated field, the tangent component is proportional to the acceleration a:

$$
E_T = \frac{qa}{c^2R}sin\theta.
$$

The energy per area per time is:

$$
\frac{dP}{d\Omega} = R^2 S = \frac{q^2a^2}{c^3}\frac{sin^2\theta}{4\pi}.
$$

The mean power of the radiation is:

$$
\begin{aligned}
\bar{P} &= \frac{1}{2}\int_0^\theta \mathrm{d}\cos\theta P_\theta,\\
&= \frac{1}{2}\frac{q^2a^2}{c^3}\int_{-1}^{1} dx(1-x^2),\\
&= \frac{2}{3}\frac{q^2a^2}{c^3}.
\end{aligned}
$$

Which is a dipole-like radiation, and most of the radiation of the accelerated particle is in th $\hat{k}$ direction.

3.4 The Relativistic Case

$$E \sim \frac{q}{c^2 R} \frac{n' \times [n' - \frac{v'}{c}] \times a'}{(1 - \frac{n'v'}{c})^2}.$$

3.4.1 Motion parallel to the acceleration

$$\hat{k} = -\hat{n}, \frac{v'}{c} = \beta' \to 1$$

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{c}{4\pi} E^2 R^2, \\ \text{where } E &= \frac{q}{c^2 R} \frac{1}{(1 - \frac{m'v'}{c})^3} a' sin\theta \hat{e}_E, \\ \text{so } \frac{dP}{d\Omega} &= \frac{q^2 a^2}{4\pi c^3} \frac{sin^2\theta}{(1 - \beta^2 cos\theta)^6}. \end{aligned}$$

The angle θ is maximum at $\sim \frac{1}{\gamma}$, and one has distortion along the directions of motion.

$$E \sim \frac{q}{c^2 R} \frac{a' sin\theta}{(1 - \beta' cos\theta^3)}.$$
$$P_{max} = \frac{1}{\gamma'^2} \frac{q^2 a^2}{c^3} \gamma'^{10} \sim \frac{q^2 a^2}{c^3} \gamma'^8.$$

3.4.2 Motion perpendicular to the acceleration

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} E^2 R^2,$$
$$E = \frac{q}{c^3 R} \frac{n' \times (n' - \frac{v'}{c}) \times a}{(1 - \frac{n\omega'}{c})^3}.$$

3.4.3 Time of emission

$$\Delta t' \sim \frac{\Delta\theta}{\omega_0} \sim \frac{1}{\gamma \omega_0}.$$
$$\Delta T \neq \Delta t'.$$
$$\frac{1}{\Delta t} = (1 - \beta)^{-1} \Delta t'.$$

The Doppler effect is given by $\omega' = \gamma\omega_0'$:

$$\frac{1}{\Delta t} = \gamma\omega_0 = \frac{1}{t'},$$
$$\omega_0 = \frac{eBc}{Ee},$$
$$E_c = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\beta'^2}}.$$

3.4.4 Chrenkov Effect

$$c' = \frac{c}{\hbar} \leq c, \text{ with } n = \sqrt{\epsilon\mu}$$
$$(\frac{1}{c'^2}\partial_t^2 - \Delta)\phi = \delta$$

If $c' < v$ there are two solutions (two emissions which the only difference is that the speed of light modifies).

4 Energy Loss

With a fast moving $E = \gamma mc^2$ there are two regimes: *free-electrons* and *bound state*. The question is how many energy is lost as the probe particle travels through the material, for this one needs to determines $\frac{dE}{dx}$. It is the same that determining how much energy is transformed.

4.1 Bohr's Formula

$$E_\perp = \gamma q \frac{b_\parallel}{(b^2+\gamma^2 v^2 t^2)^{\frac{3}{2}}}.$$
$$E_\parallel = \gamma q \frac{vt}{(b^2+\gamma^2 v^2 t^2)^{\frac{3}{2}}}.$$

$$\begin{aligned}
\Delta P_\perp &= \int_{-\infty}^{\infty} F_\perp dt, \\
&= \int_{-\infty}^{\infty} (eE_\perp) dt, \\
&= \int_{-\infty}^{\infty} \frac{dt}{(b^2+\gamma^2 v^2 t^2)^{\frac{3}{2}}}, \\
&\sim \frac{eq}{bv}, \\
&\sim \gamma^0.
\end{aligned}$$

where pulse last $t \sim \frac{b}{\gamma v} \ll 1$, and the last term keeps the trajectory straight.

4.1.1 Energy Transfered

$$\Delta E_{\perp} = \frac{(\Delta P_{\perp})^2}{2m} \sim \frac{e^2q^2}{mv^2}\frac{1}{b^2},$$

the validity of the measurement is

$$b_{min} \sim \frac{\Delta P_{\perp}}{m} \times t \sim \frac{qe}{\gamma m v^2} \sim \frac{1}{\gamma}.$$

This formula is not valid when b is very large, since the binding times is $T = \frac{2\pi}{\omega_0}$ one has $t \sim \frac{b}{\gamma v} < T \sim \frac{1}{\omega_0}$:

$$b_{max} \sim \frac{\gamma v}{\omega_0}.$$

So one has:

$$b_{min} \leq b \leq b_{max} \rightarrow \frac{eq}{\gamma m v^2} \leq b \leq \frac{\gamma v}{\omega_0}.$$

Then energy is transfered free. Finally, one can calculate the energy lost

$$\frac{dE}{dx} \sim \int [N 2\pi b db]^* \frac{(eq)^2}{mv^*}\frac{1}{b^2}, \quad (4.1)$$

$$\sim \frac{2\pi}{mv^2}(eq)^2 \int_{min}^{max} \frac{db}{b}, \quad (4.2)$$

$$\sim \frac{2\pi (eq)^2}{mv^2} \ln\left(\frac{b_{max}}{b_{min}}\right). \quad (4.3)$$

The Bohr's formulas is then

$$\frac{dE}{dx} \sim \frac{N(eq)^2}{mc^2}\gamma$$

Therefore for a **Free** charge: The time of the field is the contraction: $t_{free} < t_{binding} = \frac{1}{\omega_0}$, so $b_{max} < \frac{\gamma v}{\omega_0}$. The energy is given by 4.3.

4.1.2 Amended Formulae

From the Drude's model, one has

$$m\frac{d^2x}{dx^2} + m\Gamma\frac{dx}{dt} + m\omega_0^2 x = eE(t),$$

where $\frac{1}{\Gamma}$ is the damping time and the solutions are $x(t) = \frac{-e/m}{\omega_o^2 - \omega^2}E(\omega)E^{-i\omega t}$. The energy loss can be written as

$$\Delta W(b) = \int_{-\infty}^{\infty} F v dt = \int_{-\infty}^{\infty} eE(t)x(t)dt.$$

After a Fourier transformation

$$E(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-\omega t} x(\omega),$$

one has

$$\begin{aligned}
\Delta\omega(b) &= \frac{e}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega'(-i\omega')E(x)x(\omega') \int_{-\infty}^{\infty} dt e^{-i\omega_0}, && (4.4)\\
&= \frac{e}{2\pi} \int_{-\infty}^{\infty} d\omega(-i\omega)x(\omega)E(\omega), && (4.5)\\
&= \frac{e^2}{2\pi m} \int_{-\infty}^{\infty} d\omega |E^2(\omega)| \frac{(-i\omega)[(\omega^2 + 0 - \omega^2) + i\Gamma\omega]}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}. && (4.6)
\end{aligned}$$

Classically we only expect the real function, doing $\frac{I}{\omega_0} \ll 1$

$$\Delta\omega(b) \sim \frac{e^2}{4m} |E(\omega_0)|^2.$$

The components of the electric field may be then written as components that are proportional to the spherical Bessel functions ($k_1(\xi)$):

$$\begin{aligned}
E_\perp(\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \frac{\gamma q_b}{(b^2 + \gamma^2 v^2 t^2)^{\frac{1}{2}}},\\
&= \frac{q}{\gamma b v} \int_{-\infty}^{\infty} dt \frac{e^{i\chi t}}{(1+t^2)^{\frac{3}{2}}},\\
&= \frac{q}{bv} \xi k_1(\xi),\\
E_\parallel &= \frac{q}{bv} \int_{-\infty}^{\infty} dt \frac{e^{i\chi t}}{(1+t^2)^{\frac{3}{2}}}.
\end{aligned}$$

Rewriting (4.6) and the transfered energy (4.3) with the Bessel functions

$$\Delta\omega(b) \sim \frac{e^2 q^2}{m v^2} \frac{1}{b^2} [(\xi k_1)^2 + \frac{1}{\gamma^2} (\xi k_0(\xi))^2].$$

$$\frac{dE}{dx} \sim \frac{2\pi N e^2 q^2}{m v^2} \int_{b_{min}}^{\infty} \frac{db}{b} [(\xi k_1(\xi))^2 + \frac{1}{\gamma^2} (\xi k_0(\xi))^2],$$

where for $\xi = \frac{b\omega}{\gamma v} \ll 1$, $\xi k_1 \xi k_0 \to 1$ and $\xi \gg 1$, $k_1 \sim k_0 \sim e^{\xi} \sim e^{-\frac{b\omega}{\gamma v}}$.

4.1.3 Bethe's Amendment

The classical calculation ignores the short distances effects and the binding. Recalling the classical b_{min}, one has

$$v_\perp \tau = \frac{eq}{\gamma m v^2},$$
$$\Delta P_\perp \sim F_\perp \tau \sim \frac{eq}{bv}.$$

However, in quantum mechanics one as

$$b_{min} \sim \frac{\hbar}{\Delta P_\perp} \sim \frac{\hbar}{\gamma m v}.$$

One can then write the relation to the **binding energy** E_0:

$$\frac{b_{max}}{b_{min}} \sim \frac{\gamma v}{\omega_0} \frac{1}{\hbar/\gamma m v} = \frac{\gamma^2 m v^2}{\hbar \omega_0} = \frac{\gamma^2 m v^2}{E_0}.$$

The energy transfered is

$$\begin{aligned} \frac{dE}{dx} &\sim \frac{N(eq)^2}{mv^2} \ln \frac{b_{max}}{b_{min}}, \\ &\sim -\frac{N(eq)^2}{mv^2} \ln \frac{E_0}{\gamma^2 m v^2}, \\ &\sim \ln \gamma. \end{aligned}$$

The relation between $\frac{dE}{dx}$ and the kinetic energy $\gamma - 1 = \frac{k}{mc^2}$ can be seen at the figure 2, where the lowest point is point where k=1, and divides from the non relativistic to the ultra relativistic behavior. After the relativistic limit, one can calculate the *Plasma energy loss*:

$$\begin{aligned} \frac{dE}{dx} &\sim \frac{N(eq)^2}{mv^2} \ln \left(\frac{\gamma v}{\omega_0} \frac{1}{b_{min}} \right), \\ &\sim q^2 \frac{\omega_p^2}{v^2} \ln \left(\frac{\gamma v}{\omega_p} \right), \end{aligned}$$

where the plasma frequency is $\omega_p^2 = \frac{4\pi n e^2}{m}$.

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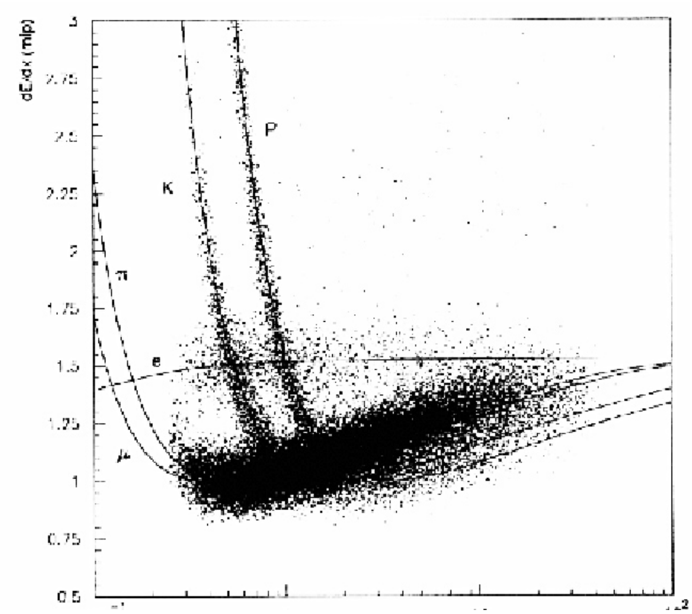

Figure 2: Relation between energy loss and the kinetic energy.

References

[1] Electrodynamics taught by Prof. Zahed, Stony Brook University (2009-2010).

[2] Classical Electrodynamics, J.D. Jackson (1999).