Parton Evolution

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December 9, 2010

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1 Parton Model

1.1 Deep Inelastic Scattering (DIS)

At a very high resolution (when the transfered momentum q^2 is large), the nucleon can be resolved as a collection of almost non-interacting point-like constituents, the partons. When the resolution scale λ of the effective probe q is smaller than the typical size of the proton (~ 1 fm), the internal structure of the proton is probed and we have the DIS regime, 1-a. Defining $Q^2 = -q^2 > 0$ as the off-shell momentum of the exchanged photon, the resolution scale is $\lambda = \frac{\hbar}{Q}$.

The electron-quark scattering, $e^-(k) + q(p_q) \rightarrow e^-(k') + q(p'_q)$, is given by the QED differential cross section (partonic massless limit $(\hat{s}+\hat{t}+\hat{u} = 0)$), where the matrix element squared for the amplitudes are

$$\sum |\mathcal{M}|^2 = 2e_q^2 e^4 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}.$$

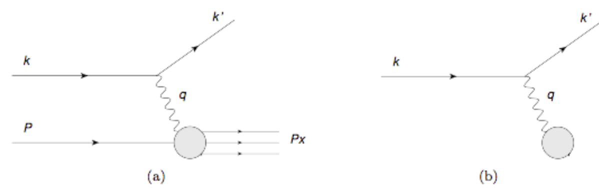

Figure 1: a)DIS, b) Elastic scattering.

In terms of the Mandelstam variables, $\hat{s} = (k + p_q)^2, \hat{t} = (k - k')^2, \hat{u} = (p_q - k')^2$. In the frame which the proton is moving very past, $P \gg M$, we can consider a simple model where the photon scatter a pointlike quark with ϵ fraction of the momentum vector $p = \xi P$. The deep inelastic kinematics are $\hat{t} = q^2, \hat{u} = \hat{s}(y-1), \hat{s} = \xi Q^2/xy$. The massless differential cross section is then

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2} \sum |\mathcal{M}|^2,$$

where, substituting the kinetic variables,

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}[1 + (1-y)^2].$$

The mass-shell constraint for the outgoing quark $p'^2_q = 0$ suggest that the structure function probes the quark with $\xi = x$. With s the square of the the center-of mass energy of the electron-hadron, the invariant $\hat{s}$ is

$$\hat{s} = (p+k)^2 \sim 2p\ .\ k \sim xs.$$

Writing $\int_0^1 dx\delta(x-\xi)=1$ we have the double cross section for the quark scattering process,

$$\frac{d^2\hat{\sigma}}{dxdQ^2}=\frac{4\pi\alpha^2}{Q^4}[1+(1-y)^2]\frac{1}{2}e_q^2\delta(x-\xi).$$

The event distribution is in the $x-Q^2$ plane. The parton distribution functions (PDFs) parametrizes the structure target as 'seen' by the virtual photon and which are not computable from first principles through perturbative calculations.

The *Bjorken limit* is definite as $Q^2, q\to\infty$, with x fixed. In this limit the structure functions obey an approximate *scaling* law, i.e. they only depend on x, not in Q^2, as we can see in figure 2. Bjorken scaling implies that the virtual photon scatters off pointlike constituents, since otherwise the structure functions would depend on the ratio Q/Q_0, some length scale.

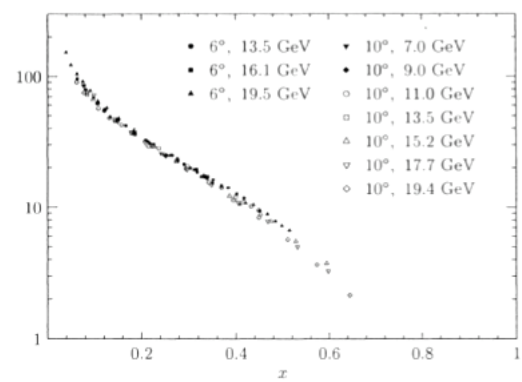

Figure 2: Bjorken scaling on e^-p DIS by the SLAC-MIT experiment, range 1 GeV^2 <Q^2 <8 GeV^2 [2].

Dividing it by $(1+(\frac{Q^2}{xs})^2)/Q^4$, we remove the dependence of the QED cross section and the result is independent of Q^2. The result is that the structure of the proton scales to an electromagnetic probe no matter how hard it is probed.

We see an anti-screening effect where the coupling constant appears strong at small momenta, behaviors called *asymptotic freedom* and belonging to a non-abelian gauge theory. These are the only field theory with asymptotic freedom behavior with interacting vector bosons which could bind the quarks. This gauge theory confirmed the Bjorken scaling, with the evolution of the coupling very slow, following a logarithmic distribution in momentum. The scaling violations corresponds to a slow evolution of the parton distributions $\mathcal{F}_i(x)$ over a logarithmic scale Q^2.

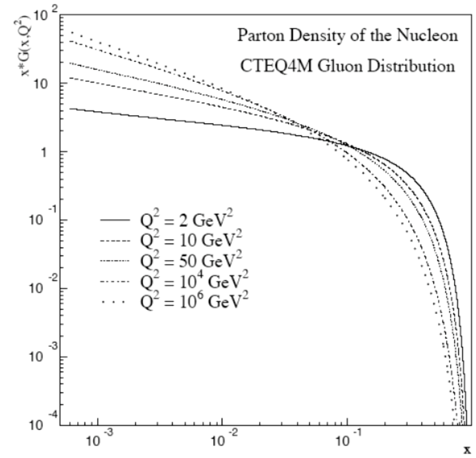

Figure 3: Gluon PDF. The Q^2 dependence is moderate except for very small x and/or very small Q^2.

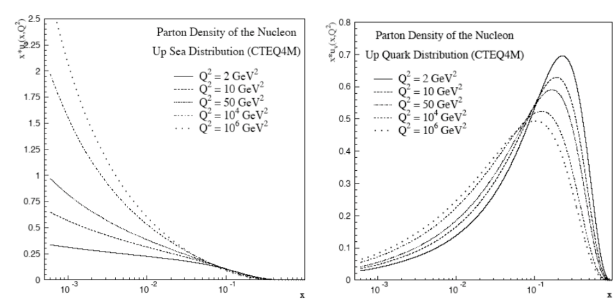

Figure 4: The up PDF from valence and sea. The Q^2 dependence is large for small x and small Q^2.

1.2 The Running Coupling

In the previous simple parton model, the structure functions scale in the asymptotic (Bjorken) limit

$Q^2 \to \infty$. In QCD, next to the leading order in α_s, this scaling is broken by logarithms of Q^2. A quark can emit a gluon and acquire large k_T with probability proportional to $\alpha_s \frac{dk_T^2}{k_T^2}$. The integral extend up the kinetic limit $k_T^2 \sim Q^2$ and gives contributions proportional to $\alpha_s Q^2$, breaking the scaling.

The screening behaviors on QED is giving by summing the higher order corrections in terms of the general form $\alpha^n[\log(Q^2/Q_0^2)]^m$ and retaining only the leading logarithm terms ($m = n$). Therefore the vacuum polarization affects the coupling in QED as

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \frac{\alpha(Q_0^2)}{3\pi} \log\left(\frac{Q^2}{Q_0}\right)},$$

the *leading logarithm approximation*.

For QCD, since gluons also emit additional gluons and similar charges attract themselves, the effect is anti-screening, the effective color charge decreases as one probes the original quark. Therefore, this phenomena of asymptotic freedom is given by the QCD running coupling,

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}(11N_c - 2N_f) \log\left(\frac{Q^2}{\mu^2}\right)}$$

where N_c is the number of color charges, N_f the number of quarks flavors and μ the renormalization scale. To make α_s independent of the renormalization scheme, we introduce a scale Λ^2_{QCD},

$$\Lambda^2_{QCD} = \mu^2 e^{\frac{-12\pi}{(11N_c - 2N_f)\alpha_s(\mu)}},$$

which experimentally is ~ 200. In this QCD scale, the coupling constant can be written as

$$\alpha_s(Q^2) = \frac{12\pi}{(11N_c - 2N_f)\log(Q^2/\Lambda^2_{QCD})},$$

where we see $\alpha_s \to 0$ when $Q^2 \to \infty$ and when $Q^2 \gg \Lambda^2_{QCD}$ we have hard interactions.

1.3 Leading Log Approximation

Collinear photon emission in QED at high energies is already associated with mass singularities and it

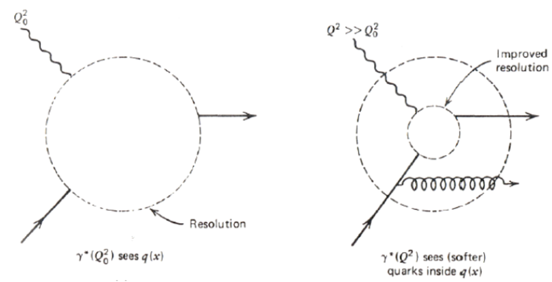

Figure 5: At Q^2 increases, it is possible to probe shorter distances. At large Q^2, the large x quarks are more likely to loose energy due to gluon radiation.

leads to an analog of parton distribution for electron. Gribov and Lipatov showed that in a field theory with dimensionless coupling α, the DIS structure functions are represented as a sum of Rutherford cross sections og lepton scattering the point-like charges particle, weighted by a parton density $\mathcal{F}_i^f(x)$.

Logarithmic deviations from the true scaling behavior had been predicted for the PDFs, reveling the internal structure of PDFs which corrections are given by $\mathcal{F}_i^f(x, \log Q^2)$. Physically: DIS with a photon with Q^2 correspond to the scattering of that virtual photon on a quark of size $1/Q$. From QCD bremsstrahlung, we have

$$d\omega \propto \alpha^2 \int^{Q^2} \frac{dk_\perp^2}{k_\perp^2},$$

therefore the collinear photon emission cost a factor not of α but α log (s/m^2), and multiple collinear photons gives contributions of order $(\alpha \log(s/m^2))^m$. therefore it makes the total probability of extra parton production large, since for $\alpha^2 \ll 1$, log Q^2 is high enough and $\omega \propto \alpha^2 \log Q^2 \sim 1$. In QCD, the corresponding factor for collinear gluon emission is $\alpha_s(Q^2)$ log $\frac{Q^2}{\mu^2}$.

In QED, when $k^2 \to 0$, we can write the expansion for $g^{\mu\nu}$ in terms of massless polarization vector and when the singular term as the photon momentum q goes on-shel is given by $\frac{-ig^{\mu\nu}}{q^2} \to \frac{+i}{q^2}\sum_i \epsilon^\mu_{Ti}\epsilon^\nu_{Ti}$ in the calculus of the amplitudes, decoupling the photon/electron emission vertex. The final cross section is giving by

$$\sigma(e^- X \to Y) = \int_0^1 \mathrm{dz}\, \frac{\alpha}{2\pi}\, \log\frac{s}{m^2}\Big[\frac{1+(1-z)^2}{z}\Big] \times \sigma(\gamma Y \to Y), = \int_0^1 \mathrm{dz}\, \mathcal{F}_\gamma(z) \times \sigma(\gamma X \to Y).$$

Considering the limit of the divergence $z = 0 = 1-x$, the soft parton emission or infrared divergent, we see it is balanced by negative contributions from diagrams with soft virtual photons. Thus, to order α, the parton distribution of electrons has the form

$$\mathcal{F}_\gamma = \delta(1-x) + \frac{\alpha}{2\pi}\, \log\, \frac{s}{m^2}\Big(\frac{1+x^2}{1-x} - A\delta(1-x)\Big), \quad (1)$$

where the first delta is the zeroth order of the expansion and A we will explicitly calculate in the following sections.

2 Evolution Equations

The hadron evolve with energy $Y = \log(1/x)$, and for some values of (Q^2, x), although the hadron is no longer perturbative, the evolution still is (except for low Q^2). Therefore, we can construct evolution equations in this two the two variables of the phase space: $\log(1/x)$ and $\log Q^2$.

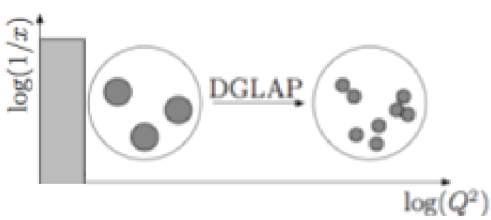

Figure 6: Proton in the phase space (Q^2, x).

In figure 6 we see that when Q^2, Y small, the proton is represented by three valence quarks and the gluon vacuum oscillations is very quickly. Increasing Q^2, the probe resolution, means to decrease the time of interaction and the probes resolves more and more of these fast fluctuations increases the number of partons seen. However, the space occupied by them decreases: at each Q^2 step the proton becomes more dilute. This evolution is described in QCD by the DGLAP equations.

The interaction described by small s is given by the partonic saturation, where the scales which separates the dilute regime from the dense is called saturation scale, $Q_s(x)$. For a spatial resolution greater than $1/Q_s$, gluons overlap the transverse plane, as described by the *Color Glass Condensate* formalism.

2.1 Perturbative Expansion of α_s

For a general quantity B, the perturbative expansion in terms of α_s is

$$B = \beta_0 + \beta_1\alpha_s + \beta_2\alpha_s^2 + \beta_3\alpha_s^3 ... \quad (2)$$

where the first power of α_s is the leading order (LO), the second the next to leading order (NLO), the third the next-to-next leading order (NNLO), etc. When $Q^2 \sim \Lambda^2_{QCD}$, we have soft processes and $\alpha_s \to 1$, the expansion does not converge.

2.2 Gluon Spitting Function

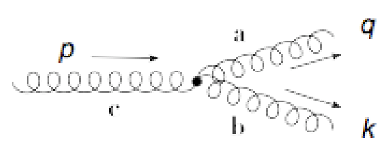

Figure 7: Three-gluon vertex.

The element matrix (Feynman rules) for 7 is

$$\mathcal{M} = -igf^{abc}[\epsilon^*(q)\epsilon(p)\epsilon^*(k)(p+q) + + \epsilon^*(q)\epsilon^*(k)\epsilon(p)(-q+k) - - \epsilon^*(k)\epsilon(p)\epsilon^*(q)(k+p).$$

The color factor of the squared average amplitude

$$\frac{1}{d_a}\sum_{b,c} f^{abc} f^{*abc} = C_A = 3,$$

where C_A is the Casimir for the adjoint representation of SU(3) (gluon). We choose the gluon (left, right) polarization

$$\epsilon_\perp^R = \frac{1}{\sqrt{2}}(-1,-i), \epsilon_\perp^L = \frac{1}{\sqrt{2}}(1,-i).$$

The squared amplitude is then

$$\mathcal{M}^2 = \frac{12g^2p_\perp^2}{z(1-z)}\frac{z^2(1-(1-z)z)+(1-z)^2(z+(1-z)^2}{z(1-z)}.$$

In the light-cone basis, the kinematics of a a real gluon can be written as

$$\begin{aligned} p &= (\sqrt{2}p, 0.0), \\ q &\sim (\sqrt{2}zp - \frac{\sqrt{2}p_\perp^2}{4zp}, \frac{\sqrt{2}p_\perp^2}{4zp}, p_\perp, 0), \\ k &\sim (\sqrt{2}(1-z)p + \frac{\sqrt{2}p_\perp^2}{4zp}, -\frac{\sqrt{2}p_\perp^2}{4zp}, -p_\perp, 0). \end{aligned}$$

Using these kinematic relations we have

$$\begin{aligned} & \frac{1}{4\pi}\sum|\mathcal{M}|^2 \propto P_{g\leftarrow g}(z) \\ = \; & 2C_A\Big[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \\ + \; & A\delta(1-z)\Big] \end{aligned}$$

Calculating the Normalization

We know calculate the normalization A from the last equation. We take into account the process where the gluons remains with its identity and normalize through defining a distribution that can be integrated by subtracting a delta function. We define a function that agrees to $1/(1-z)$ for all values of $x<1$,

$$\frac{1}{(1-z)_+} = \frac{1}{(1-z)}, \forall z \in [0,1[,$$

the integral of this distribution with any smooth function $f(x)$ gives the new distribution

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{1-z}.$$

In the QED process this normalization was $A=\frac{2}{3}$. For our gluon splitting function we are going to use the relation (4.93) of [4]. The leading-order DGLAP splitting function $P_{a\leftarrow b}(x)$ has an attractive interpretation as the probabilities of finding a parton of type b with a fraction x of the longitudinal momentum and a transverse momentum squared much less than μ^2. The interpretation as probabilities implies the sum rules in the leading order,

$$\int_0^1 \text{dx } P_{q\leftarrow q}(x) = 0, \tag{3}$$

$$\int_0^1 \text{dx x } \Big[P_{q\leftarrow q}(x) + P_{g\leftarrow q}(x)\Big] = 0, \tag{4}$$

$$\int_0^1 \text{dx x } \Big[2N_f P_{q\leftarrow g}(x) + P_{g\leftarrow g}(x)\Big] = 0. \tag{5}$$

We will use the last one to find the overall normalization, A, on 3, substituting the know splitting functions. For the first term,

$$\begin{aligned} &\int_0^1 \text{dx x } \Big[2N_f P_{q\leftarrow g}(x)\Big] = \\ &\int_0^1 \text{dx x } \Big[2N_f \text{ Tr } [x^2+(1-x)^2]\Big] = \\ &N_f \int_0^1 \text{dx x } \Big[x^2+(1-x)^2\Big] = \frac{N_f}{3}. \end{aligned}$$

The second therm $(x = 1-z)$,

$$\begin{aligned} &\int_0^1 \text{dx x } [P_{g\leftarrow g}(x)] = \\ &2C_A \int_0^1 \text{dx x } \Big[\frac{x}{1-x} + \frac{1-x}{x} + x(1-x)\Big] = \\ &6\int_0^1 \text{dx x } \Big[\frac{x-1}{1-x} + \frac{x-1}{x} + x(1-x)\Big] = \\ &6\Big(-\frac{11}{12}\Big). \end{aligned}$$

Summing up both terms, we find $6A = 6[\frac{11}{2} - \frac{N_f}{3}]$. Finally, the leading-order corrected splitting function, 3, is then given by

$$P^0_{g\leftarrow g}(z) = 6\Big[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) + \Big(\frac{11}{12} - \frac{N_f}{18}\Big)\delta(1-z)\Big)\Big],$$

.

In the language of the expansion on 2, the (gluon) splittings functions can be written as

$$P_{g\leftarrow g}(z,\alpha_s) = P^0_{g\leftarrow g} + \frac{\alpha_s}{2\pi} P^1_{\leftarrow gg}(z) + ...$$

2.3 Proprieties of Splitting Functions

$$\begin{aligned} P_{q\leftarrow q}(z) &= P_{q\leftarrow q}(1-z), \\ P_{q\leftarrow g}(z) &= P_{q\leftarrow g}(1-z), \\ P_{g\leftarrow g}(z) &= P_{g\leftarrow g}(1-z). \end{aligned}$$

Because of the charge conjugation invariance and $SU(N_f)$ flavor symmetry,

$$\begin{aligned} P_{q_i\leftarrow q_j}(z) &= P_{\bar{q}_i\leftarrow \bar{q}_j} \\ P_{q_i\leftarrow \bar{q}_j}(z) &= P_{\bar{q}_i\leftarrow q_j} \\ P_{q_i\leftarrow g}(z) &= P_{\bar{q}_i\leftarrow g} = P_{q\leftarrow g} \\ P_{g\leftarrow q_i}(z) &= P_{g\leftarrow \bar{q}_i} = P_{g\leftarrow q} \end{aligned}$$

2.4 Altarelli-Parisi Equations

Together with the other splitting functions we can write explicitly the Altarelli-Parisi equations which describes the coupled evolution of the PDFs, $\mathcal{F}_g(x,Q), \mathcal{F}_f(x,Q), \mathcal{F}_{\bar{f}}(x,Q)$, for each flavor of quark and antiquarks (treated as massless at Q),

$$\frac{d}{d\log Q}\mathcal{F}_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\left[P_{g\leftarrow q}\sum_f \Big(\mathcal{F}_f(\frac{x}{z},Q) + \mathcal{F}_{\bar{f}}(\frac{x}{z},Q)\Big) + P_{g\leftarrow g}(z)\mathcal{F}_g(\frac{x}{z},Q)\right],$$

$$\frac{d}{d\log Q}\mathcal{F}_f(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\left[P_{q\leftarrow q}\mathcal{F}_f(\frac{x}{z},Q) + P_{q\leftarrow g}(z)\mathcal{F}_g(\frac{x}{z},Q)\right],$$

$$\frac{d}{d\log Q}\mathcal{F}_{\bar{f}}(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\left[P_{q\leftarrow q}\mathcal{F}_{\bar{f}}(\frac{x}{z},Q) + P_{q\leftarrow g}(z)\mathcal{F}_g(\frac{x}{z},Q)\right],$$

Generally, the DGLAP equation is a $(2N_f+1)$-dimensional matrix equation in the space of quarks, antiquarks and gluons.

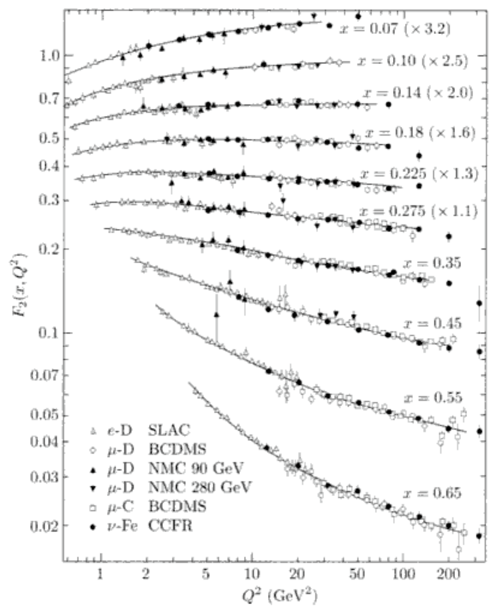

Figure 8: Dependence on Q^2 of the quark PDF in a DIS electron-proton. The curves are the results from Altarelli-Parisi equations.

References

[1] Particle Data Group, 2010, `http://pdg.ihep.su/2010/figures/figures.html`.

[2] POUCHER, et al., Phys Rev Lett 32, 118 (1974).

[3] PESKIN, M.; Introduction to Quantum Field Theory.

[4] WEINBERG, S; Quantum Field Theory II.

[5] ALTARELLI, G. and PARISI, G.; Nucl. Phys. B126 (1977), p. 298.

[6] GLAZOV, Sasha and H1 COLLABORATION. Measurement of DIS cross section at HERA. Braz. J. Phys, 2007, vol.37, n.2c, pp. 793-797.