Introduction to Group Theory for Physicists

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Preface

These notes started after a great course in group theory by Dr. Van Nieuwenhuizen [8] and were constructed mainly following Georgi's book [3], and other classical references. The purpose was merely educative. This book is made by a graduate student to other graduate students. I had a lot of fun putting together my readings and calculations and I hope it can be useful for someone else.

Marina von Steinkirch,
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Contents

Chapter 1

Finite Groups

A *finite group* is a group with finite number of elements, which is called the order of the group. A *group* G is a set of elements, $g \in G$, which under some operation rules follows the common proprieties

1. Closure: g_1 and $g_2 \in G$, then $g_1 g_2 \in G$.
2. Associativity: $g_1(g_2 g_3) = (g_1 g_2)g_3$.
3. Inverse element: for every $g \in G$ there is an inverse $g^{-1} \in G$, and $g^{-1}g = gg^{-1} = e$.
4. Identity Element: every groups contains $e \in G$, and $eg = ge = g$.

1.1 Subgroups and Definitions

A *subgroup* H of a group G is a set of elements of G that for any given $g_1, g_2 \in H$ and the multiplication $g_1 g_2 \in H, G$, one has again the previous four group proprieties. For example, S_3 has Z_3 as a subgroup[1]. The trivial subgroups are the identity e and the group G.

Cosets

The *right coset* of the subgroup H in G is a set of elements formed by the action of H on the left of a element of $g \in G$, i.e. Hg. The *left coset* is gH. If each coset has $[H]$ elements[2] and for two cosets of the same group one has $gH_1 = gH_2$, then $H_1 = H_2$, meaning that cosets do not overlap.

[1] These groups will be defined on the text, and they are quickly summarized on table A.1, in the end of this notes.

[2] $[G]$ is the notation for number of elements (order) of the group G.

Lagrange's Theorem

The order of the coset H, $[H]$ is a divisor of $[G]$,

$$[G] = [H] \times n_{cosets},$$

where n_{cosets} is the number of cosets on G.

For example, the permutation group S_3 has order $N! = 3! = 6$, consequently it can only have subgroups of order 1, 2, 3 and 6. Another direct consequence is that groups of prime order have no *proper* (non-trivial) subgroups, i.e. prime groups only have the trivial $H = e$ and $H = H$ subgroups.

Invariant or Normal or Self-conjugated Subgroup[3]

If for every element of the group, $g \in G$, one has the equality $gH = Hg$, i.e. the right coset is equal to the left coset, the subgroup is *invariant*. The trivial e and G are invariant subgroups of every group.

If H is a invariant coset of a group, we can see the coset-space as a group, regarding each coset as a element of the space. The coset space G/H, which is the sets of cosets, is a factor group given by the factor of G by H.

Conjugate Classes

Classes are the set of elements (not necessary a subgroup) of a group G that obey $g^{-1}Sg = S$, for all $g \in G$. The term gSg^{-1} is the conjugate of S. For a finite group, the number of classes of a group is equal to the number of *irreducible representations (irreps)*. For example, the conjugate classes of S_3 are $[e, (a_1, a_2), (a_3, a_4, a_5)]$.

An invariant subgroup is composed of the union of all (entire) classes of G. Conversely, a subgroup of entire classes is an invariant of the group.

Equivalence Relations

The *equivalence relations* between two sets (which can be classes) are given by

1. Reflexivity: $a \sim a$.
2. Symmetry: if $a \sim b$, then $b \sim a$.
3. Transitivity: if $a \sim c$ and $b \sim c$, then $a \sim b$.

[3]We shall use the term invariant in this text.

Quotient Group

A *quotient group* is a group obtained by identifying elements of a larger group using an equivalence relation. The resulting quotient is written G/N^4, where G is the original group and N is the invariant subgroup.

The set of cosets G/H can be endowed with a group structure by a suitable definition of two cosets, $(g_1H)(g_2H) = g_1g_2H$, where g_1g_2 is a new coset. A group G is a direct product of its subgroups A and B written as $G = A \times B$ if

1. All elements of A commute to B.

2. Every element of G can be written in a unique way as $g = ab$ with $a \in A, b \in B$.

3. Both A and B are invariant subgroups of G.

Center of a Group $Z(G)$

The *center* of a group G is the set of elements of G that commutes with all elements of this group. The center can be trivial consisting only of e **or** G. The center forms an *abelian* [5] *invariant subgroup* and the whole group G is abelian only if $Z(G) = G$.

For example, for the Lie group SU(N), the center is isomorphic to the cyclic group Z_n, i.e. the largest group of commuting elements of SU(N) is $\simeq Z_n$. For instance, for SU(3), the center is the three matrices 3×3, with $\mathrm{diag}(1,1,1)$, $\mathrm{diag}(e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}})$ and $\mathrm{diag}(e^{\frac{4\pi i}{3}}, e^{\frac{4\pi i}{3}}, e^{\frac{4\pi i}{3}})$, which clearly is a phase and has determinant equals to one. On the another hand, the center of U(N) is an abelian invariant subgroup and for this reason the unitary group is not *semi-simple*[6].

Concerning finite groups, the center is isomorphic to the trivial group for $S_n, N \geq 3$ and $A_n, N \geq 4$.

Centralizer of an Element of a Group $c_G(a)$

The centralizer of a, $c_G(a)$ is a new subgroup in G formed by $ga = ag$, i.e. the set of elements of G which commutes with a.

[4] This is pronounced G mod N.

[5] An abelian group is one which the multiplication law is commutative $g_1g_2 = g_2g_1$.

[6] We will see that semi-simple Lie groups are direct sum of simple Lie algebras, i.e. non-abelian Lie algebras.

An element a of G lies in the center $Z(G)$ of G if and only if its conjugacy class has only one element, a itself. The centralizer is the largest subgroup of G having a as it center and the order of the centralizer is related to G by $[G] = [c_G(a)] \times [\text{class}(a)]$.

Commutator Subgroup $C(G)$

The *commutator group* is the group generated from all commutators of the group. For elements g_1 and g_2 of a group G, the commutator is defined as

$$[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2.$$

This commutator is equal to the identity element e if $g_1 g_2 = g_2 g_1$, that is, if g_1 and g_2 commute. However, in general, $g_1 g_2 = g_2 g_1 [g_1, g_2]$.

From $g_1 g_2 g_1^{-1} g_2^{-1}$ one forms finite products and generates an invariant subgroup, where the invariance can be proved by inserting a unit on $g(g_1 g_2 g_1^{-1} g_2^{-1}) g^{-1} = g' g'^{-1}$.

The commutator group is the smallest invariant subgroup of G such that $G/C(G)$ is abelian, which means that the large the commutator subgroup is, the "less abelian" the group is. For example, the commutator subgroup of S_n is A_n.

1.2 Representations

A *representation* is a mapping $D(g)$ of G onto a set, respecting the following rules:

1. $D(e) = 1$ is the identity operator.

2. $D(g_1) D(g_2) = D(g_1 g_2)$.

The dimension of a representation is the dimension of the space on where it acts. A representation is *faithful* when for $D(g_1) \neq D(g_2)$, $g_1 \neq g_2$, for all g_1, g_2.

The Schur's Lemmas

Concerning to representation theory of groups, the *Schur's Lemma* are

1. If $D_1(g) A = A D_2(g)$ or $A^{-1} D_1(g) A = D_2(g)$, $\forall g \in G$, where $D_1(g)$ and D_2 are inequivalent irreps, then $A = 0$.

2. If $D(g)A = AD(g)$ or $A^{-1}D(g)A = D(g)$, $\forall g \in G$, where $D(g)$ is a finite dimensional irrep, then $A \propto I$. In other words, any matrix A commutes with all matrices $D(g)$ if it is proportional to the unitary matrix. One consequence is that the form of the basis of an irrep is unique, $\forall g \in G$.

Unitary Representation

A representation is unitary if all matrices $D(g)$ are unitary. Every representation of a compact finite group is equivalent to a unitary representation, $D^\dagger = D^{-1}$.

Proof. Let $D(g)$ be a representation of a finite group G. Constructing

$$S = \sum_{g \in G} D(g)^\dagger D(g),$$

it can be diagonalized with non-negative eigenvalues $S = U^{-1}dU$, with d diagonal. One then makes $x = S^{1/2} = U^{-1}\sqrt{d}U$, from this, one defines the unitary $D'(g) = xD(g)x^{-1}$. Finally, one has $D'(g)^\dagger D'(g) = x^{-1}D(g)^\dagger SD(g)x^{-1} = S$. □

Reducible and Irreducible Representation

A representation is reducible if it has an invariant subspace, which means that an action of a $D(g)$ on any vector in the subspace is still a subspace, for example by using a projector on the regular representation (such as $PD(g) = P, \forall g$). An irrep is a representation that has no nontrivial invariant subspaces.

Every representation of a finite group is completely reducible and it is equivalent to the block diagonal form. For this reason, any representation of a finite or semi-simple group can breaks up into a direct sum of irreps. One can always construct a new representation by a transformation

$$D(g) \to D'(g) = A^{-1}D(g)A, \tag{1.2.1}$$

where $D(g)$ and $D'(g)$ are equivalent representations, only differing by choice of basis. This new representation can be made diagonal, with blocks representing its irreps. The criterium of diagonalization of a matrix $D(g)$ is that it commutes to $D(g)^\dagger$.

Characters

The *character* of a representation of a group G is given by the trace of this representation. The application of the theory of characters is given by *orthogonality relation* for groups,

$$\sum_g D^i(g^{-1})^\nu_\mu D^j(g)^\sigma_\rho = \delta^\sigma_\mu \delta^\rho_\nu \delta^{ij} \frac{[G]}{n_i}, \tag{1.2.2}$$

where n_i is the dimension of the representation $D^i(g)$. An alternative way of writing (1.2.2) is

$$\sum_g D^{*i}(g)^\nu_\mu D^j(g)^\sigma_\rho = \delta^\sigma_\mu \delta^\rho_\nu \delta^{ij} \frac{[G]}{n_i}. \tag{1.2.3}$$

The character on this representation is given by

$$\chi^i(g) = D^i(g)^\mu_\mu, \tag{1.2.4}$$

and using back (1.2.2), $\delta^\sigma_\mu \delta^\rho_\nu = \delta_{ij}\delta_{ij} = \delta_{ii} = n_i$, one can check that characters of irreps are orthonormal

$$\frac{1}{[G]} \sum_g \chi^i(g)\chi^{j*}(g) = \delta^{ij}. \tag{1.2.5}$$

Because of the cyclic propriety of the trace, χ is the same for all equivalent representations, given by (1.2.1). The character is also the same for conjugate elements tr $[D(hgh^{-1})]$ = tr $[D(h)D(g)D(h)^{-1}]$ = tr $D(g)$. Therefore, we just proved the statement that the number of irreps is equal to the number of conjugate classes.

For finite groups, one can construct a character table of a group:

1. The number of irreps are equal to the number of conjugacy classes, therefore, one can label the table by the irreps $D_1(g), D_2(g), ...$ and the conjugacy classes of elements of this group.

2. In the case of an abelian groups, all irreps are one-dimensional and from Schur's theorem, all matrices are diagonal. If the representation is greater than one-dimensional, the representation is reducible.

3. To complete the columns, one can use the that (from the orthogonally relation), $[G] = \sum_c n_c$, where the sum is over all classes c and n_c is the dimension of the classes.

Regular Representations

The *Caley's theorem* says that there is an isomorphism between the group G and a subgroup of the symmetric group $S_{[G]}$. The $[G] \times [G]$ permutation matrices $D(g)$ form a representation of the group, the *regular representation*. The dimension of the regular representation is the order of the group.This representation can be decomposed on N blocks [7],

$$D^{reg} = D^p \otimes ... \otimes D^p. \tag{1.2.6}$$

Each irrep appears in the regular representation a number times equal to its dimension, e.g. if the dimension of a D^{p_1} is 2, then D^{reg} has the two blocks $D^{p_1} \otimes D^{p_1}$.

One can take the trace in each block to find the character of the regular representation

$$\chi^{reg}(g) = a^1\chi^p + a^2\chi^p ... \tag{1.2.7}$$

$$= \sum^p a_p\chi^p(g), \tag{1.2.8}$$

giving the important result,

$$\chi^{reg}(g) = [G], \text{ if } g = e;$$
$$\chi^{reg}(g) = 0, \text{ otherwise.}$$

Therefore, it is possible to decompose (1.2.6) as

$$D^{reg} = \sum_{\oplus} a^p D^p, \tag{1.2.9}$$

where a^p is giving by (1.2.8), thus

$$a^p = \frac{1}{[G]} \sum_g \chi^{reg}(g)\chi^p(g^{-1}). \tag{1.2.10}$$

One consequence of the orthogonality relation is that the order of the group G is the sum of the square of all irreps (or classes) of this group,

$$[G] = \sum_p n_p^2, \tag{1.2.11}$$

where n_p is the dimension of the of each irrep. The number of one-dimensional irreps of a finite group is equal to the order of $G/[C(G)]$, where $C(G)$ is the commutator subgroup.

[7] The index p denotes irreps.

1.3 Reality of Irreducible Representations

For compact groups, irreps can be classified into real, pseudo-real and complex using the equivalence equation (1.2.1), with the following definitions:

1. An irrep $D(g)$ is real if for some S, $D(g)$ can be made real by $SD(g)S^{-1} = D(g)^{real}$. In this case S is symmetric. The criterium using character is that $\sum_g \chi(g^2) = [G]$,

2. An irrep is pseudoreal if on making $SD(g)S^{-1} = D(g)^{complex}$, the equivalent $D(g)^{complex}$ is complex. In this case, S is anti-symmetric. The character criterion is $\sum_g \chi(g^2) = 0$.

3. An irrep is complex if one cannot find a $D(g)'$ which is equivalent to $D(g)$. The character criterium them gives $\sum_g \chi(g^2) = -[G]$

Example: C_3

For the cyclic group $C_3 = (e, a, a^2)$, $a^3 = e$, one representations is given by $e = 1, a = e^{\frac{2\pi i}{3}}, a^2 = e^{\frac{4\pi i}{3}}$. Calculating the characters,

$$\sum_g \chi(g^2) = \chi(1) + \chi(e^{\frac{4\pi i}{3}}) + \chi(e^{\frac{8\pi i}{3}}) = 0$$

thus the representation is pseudo-real.

1.4 Transformation Groups

The *transformation groups* are the groups that describe symmetries of objects. For example, in a quantum mechanics system, a transformation takes the *Hilbert space* into an equivalent one. For each group element g, there is a unitary $D(g)$ that maps the Hilbert space into an equivalent representation and these unitary operators form a representation of the symmetric group for the Hilbert space (and the new states have the same eigenvalue, $[H, D(g)] = 0$). Again here, the dimension of a representation is the dimension of the space on which it acts.

Permutation Groups, S_n

Any element of a *permutation (or symmetric)* group S_n can be written in terms of cycles where a cycle is a cyclic permutation of a subset. Permutations are even or odd if they contain even or odd numbers of two-cycles.

For example, S_3 is the permutation on 3 objects, $[e, a_1 = (1,2,3), a_2 = (3,2,1), a_3 = (1,2), a_4 = (2,3), a_5 = (3,1)]$, and $(123) \rightarrow (12)(23)$ is even.

The order of S_n is $N!$. There is a simple N-dimensional representation of S_n called the *defining representation*, where permuted objects are the basis of a N vector space. Permutation groups appears on the relation of the *special orthogonal groups* as

$$S_n = \frac{SO(n+1)}{SO(n)}.$$

Dihedral Group, D_{2n}

The *dihedral group* is the group symmetry of a regular polygon, and the group has two basic transformations (called isometries), **rotation** , with det=1,

$$g_k = \begin{pmatrix} cos\frac{2\pi k}{n} & -sin\frac{2\pi k}{n} \\ sin\frac{2\pi k}{n} & cos\frac{2\pi k}{n} \end{pmatrix},$$

and **reflection**, with det= -1,

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The dihedral group is non-abelian[8] if $N > 2$. A polyhedra in according to its dimension is categorized on table 1.1. The transformation groups acts on vertices, half of vertices and cross-lines of symmetrical objects.

Dimension	Object
d=0	Point
d=1	Line
d=2	D_{2N}
d=3	Polyhedra
d=4	Polytopes

Table 1.1: Geometric objects related to their dimensions.

[8] For non-abelian groups, at least some of the representations must be in a matrix form, since only matrices can reproduce non-abelian multiplication law.

Cyclic Groups, Z_n

A *cyclic group* is a group that can be generated by a single element, g (called the generator of the group), such that when written multiplicatively, every element of the group is a power of g. For example, for the group Z_3, described on section 1.3, the regular representation is

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, D(a^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

One can see that for both, this above representation and the regular representation of Z_n, they can be made cyclic by the multiplication of the generator.

Another example is the parity operator on quantum mechanics $[e, p]$, where $p^2 = 1$. This group is Z_2 and has two irreps, the trivial $D(p) = 1$ and one in which $D(e) = 1, D(p) = -1$. On one-dimensional potentials that are symmetric on $x = 0$, their eigenvalues are either symmetric or anti-symmetric under $x \to -x$, corresponding to those two irreps respectively.

Alternating Groups, A_n

The alternating group is the group of even permutations of a finite set. It is the commutator subgroup of S_n such that $S_n/A_n = S_2 = C_2$.

Chapter 2

Lie Groups

A *Lie Group* is a group which is also a manifold, thus there is a small neighborhood around the identity which looks like a piece of $\mathbb{R}^N$, where N is the dimension of the group. The coordinate unit vector T_a are the elements of the *Lie algebra* and an arbitrary element g close to the identity can be always expanded into these coordinates as in (2.1.1). A Lie group can have several disconnected pieces and the Lie algebra specifies only the connected pieces containing the identity.

2.1 Lie Algebras

In *compact*[1] *Lie Algebras*, that are the interest here, the number of generators T^a is finite and the structure constant on the (2.1.2) are real and antisymmetric. Any infinitesimal group element g close to the identity can be written as

$$g(a) = 1 + ia^a T_a + \mathcal{O}(a^2), \tag{2.1.1}$$

where the multiplication of two group elements $g(a), g(b)$,

$$[T^a, T^b] = if^{abc}T^c, \tag{2.1.2}$$

[1]In general, a *set* is compact if every infinite subset of it contains a sequence which converges to an element of the same set. A *closed* group, whose parameters vary over a finite range, is compact, every continuous function defined on a compact set is bounded. It defines a connected algebras, for example SO(4) is compact but SO(3,1) is not, or for instance, a region of finite extension in an euclidian space is compact. The integral of a continuous function over the compact group is well defined and every representation of a compact group is equivalent to a unitary representation.

is given by the non-abelian generators commutation relation. All the four proprieties defined for finite groups verify for this continuous definition of elements of groups. For instance, the **closure** is proved by making

$$e^{i\lambda_1^a T_a} e^{i\lambda_2^b T_b} = e^{i\lambda_3^c T_c} = e^{i(\lambda_1^a T_a + \lambda_2^b T_b + \frac{1}{2}[\lambda_1^a T_a \lambda_2^b T_b])}.$$

U(N)	Group of all $N \times N$ unitary matrices.
Lie algebra	Set of $N^2 - 1$ hermitians $N \times N$-matrices.
SO(N)	Group of all $N \times N$ orthogonal matrices.
Lie algebra	Set of $2N^2 \pm N$ complex antisymmetric $N \times N$ matrices.

Table 2.1: Distinction among the elements of the group and the elements of the representation (the generators), for Lie groups.

Semi-Simple Lie Algebra

If one of the generators of the algebra commutes with all the others, it generates an independent continuous abelian group $\psi \to e^{i\theta}\psi$, called U(1). If the algebra contains this elements it is *semi-simple.* This is the case of algebras without *abelian* invariant subalgebras, but constructed by putting simple algebras together. Invariant subalgebras (as defined before) are sets of generators that goes into themselves under commutation. A mathematical way of expressing it is by means of *ideals.* A normal/invariant subgroup is generated by an invariant subalgebra, or ideal, I, where for any element of the algebra, L, $[I, L] \subset I$. A semi-simple group has no abelian ideals.

Non semi-simple Algebra	Contains ideals I where $[I, L] \subset I$
Semi-simple Algebra	All ideals I are non-abelian $[I, L] \neq 0 \subset I$
Simple Algebra	No Ideals, only trivial invariant subalgebra.

Table 2.2: Definition algebras in terms of ideals.

A generic element of U(1) is $e^{i\theta}$ and any irrep is a 1×1 complex matrix, which is a complex number. The representation is determined by a charge q, with the group element $g = e^{i\theta}$ represented by $e^{iq\theta}$. In a lagrangian with U(1) symmetry, each term on the lagrangian must have the charges add up to zero.

Every complex semi-simple Lie algebra has precisely one compact real form. In semi-simple Lie algebras every representation of finite degrees is fully reducible. The necessary and sufficient condition for a algebra to be semi-simple is that the *Killing metric* is non-singular, (2.4.4), $g_{\alpha\beta} \neq 0$, i.e. it has an inverse $g^{\alpha\beta} g_{\beta\nu} = \delta_{\alpha\nu}$. If $g_{\alpha\beta}$ is **negative** definite, the algebra is compact, thus it can be rescaled in a suitable basis $g^{\alpha\beta} = -\delta^{\alpha\beta}$.

Non Semi-Simple Lie Algebra

A non semi-simple Lie algebra A is a direct sum of a solvable Lie algebra (P) and a semi-simple Lie algebra (S). The definition of solvable Lie algebra is giving by the relation of commutation of the generators,

$$[A, A] = A^1$$

$$[A^1, A^1] = A^K,$$

$$....$$

where if at some point one finds $[A^n, A^n] = A^{n+1} = 0$, the algebra is solvable.

Example: The Poincare Group

Recalling the generators of the Poincare group SO(1,3), one has the semi-simple (simple + abelian) and the solvable:

$$\begin{aligned} [M, M] &= M, \text{ Simple}, \\ [P, P] &= 0, \text{ Abelian}, \\ [M, P] &= P, \text{ Solvable sector}. \end{aligned}$$

Simple Lie Algebra

A simple Lie algebra contains no deals (it cannot be divided into two mutually commuting sets). It has no nontrivial invariant subalgebras.

The generators are split into a set $\{H\}$, which commutes to each other, and the rest, $\{E\}$, which are the generalized raising and lowering operators. The classification of the algebra is specified by the number of simple roots, whose lengths and scalar products are restricted and can be summarized by the *Cartan matrix* and the *Dynkin diagrams*.

For simple compact Lie groups, the Lie algebra gives an unique simply connected group and any other connected group with same algebra must be a quotient of this group over a discrete identification map. For instance, the Lie algebra of rotations SU(2) and the group SO(3) have the same Lie algebra but they differ by 2π, which is represented in SU(2) by diag (-1,1)[2].

The condition that a Lie Algebra is compact and simple restricts it to 4 infinity families (A_n, B_n, C_n, D_n) and 5 exceptions (G_5, F_4, E_6, E_7, E_8). The families are based on the following transformations:

Unitary Transformations of N-dimensional vectors For η, ξ, N-vectors, with linear transformations $\eta_a \to U_{ab}\eta_b$ and $\xi_a \to U_{ab}\xi_b$, this subgroup preserves the unitarity of these transformations, i.e. preserves $\eta_a^* \xi^a$. The pure phase transformation $\xi_a e^{i\alpha}\xi_a$ is removed to form SU(N), consisting of all $N \times N$ hermitian matrices satisfying det(U)=1. The $N^2 - 1$ generators of the group are the $N \times N$ matrices T^a under the condition $\mathrm{tr}[T^a] = 0$.

Orthogonal Transformations of 2N-dimensional vectors The subgroup of orthogonal $2N \times 2N$ transformations that preserves the symmetric inner product: $\eta_a E_{ab} \xi_b$ with $E_{ab} = \delta_{ab}$, which is the rotation group in 2N dimensions, SO(n) (we will use $n = 2N$ and $n = 2N + 1$). There is an independent rotation to each plane in n dimensions, thus the number of generators are $\frac{n(n-1)}{2}$, or $2N^2 \pm N$.

Symplectic Transformations of N-dimensional vectors The subgroup of unitary $N \times N$ transformations where for N even, it preserves the antisymmetric inner product $\eta_a E_{ab} \xi_b$, where

$$E_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The elements of the matrix are $\frac{N}{2} \times \frac{N}{2}$ blocks, defining the symplectic group Sp(n), with $\frac{n(n+1)}{2}$ or $2N^2 + N$ generators.

2.2 Representations

Matrices associated with elements of a group are a *representation* of this group. Every group has a representation that is singlet or trivial, in which

[2] $SO(3)/SU(2) \simeq Z_2$.

Group	Number of generators	Rule for Dimension
SU(2)	3	
SU(3)	8	
SU(4)	15	$N \to N^2 - 1$
SU(5)	24	
SU(6)	35	
SO(3)	3	
SO(5)	10	$2N + 1 \to 2N^2 + N$
SO(4)	6	
SO(6)	15	$2N \to 2N^2 - N$
SO(10)	45	
Sp(2)	3	
Sp(4)	10	$2N \to 2N^2 + N$
Sp(6)	21	

Table 2.3: Number of generators for the compact and simple families on the Lie Algebra.

$D(g)$ is the 1×1 matrix for each g and $T_a = 0$. The invariance of a lagrangian under a symmetry is equivalent to the requirement that the lagrangian transforms under the singlet representation.

If the Lie algebra is semi-simple, the matrices of the representation, T_r^a, are traceless and the trace of two generator matrices are positive definite given by

$$\text{tr } [T_r^a, T_r^b] = D^{ab}.$$

Choosing a basis for T^a which has $D^{ab} \propto \mathcal{I}$ for one representation means that for all representations one has

$$\text{tr } [T_r^a, T_r^b] = \alpha \delta_{ab}. \tag{2.2.1}$$

From the commutation relations (2.1.2), one can write the anti-symmetric structure constant as

$$f^{abc} = -\frac{i}{C(r)} \text{ tr } \Big[[T_r^a, T_r^b] T_r^c \Big], \tag{2.2.2}$$

where $C(r)$ is the quadratic Casimir operator, defined on section 2.7. For each irrep r of G, there will be a conjugate representation $\bar{r}$ given by

$$T_{\bar{r}}^a = -(T_r^a)^* = -(T_r^a)^T, \tag{2.2.3}$$

if $\bar{r} \sim r$ then $T^a_{\bar{r}} = UT^a_r U^\dagger$ and the representation is real or pseudo-real, if there is no such equivalence, the representation is complex.

The two most important irreducible representations are the *fundamental* and the *adjoint* representations:

Fundamental In SU(N) the basic irrep is the N-dimensional complex vector, and for $N > 2$, this irrep is complex. In SO(N) it is real and in Sp(N) it is pseudo-real.

Adjoint It is the representation of the generators, $[r] = [G]$, and the representation's matrices are given by the structure constants $(T_a)_{bc} = -f^a_{bc}$ where $([T^b_G, T^c_G])_{ae} = if^{bcd}(T^d_G)_{ae}$. Since the structure constants are real and anti-symmetric, this irrep is always real.

2.3 The Defining Representation

A subset of commuting hermitian generators which is as large as possible is called the *Cartan subalgebra*, and it is always unique. The basis are called defining (fundamental) representation and are given by the colum -vectors $(1, 0, 0, .., 0)_N$, etc. In an irrep, D, there will be a number of hermitian generators, H_i for $i = 1$ to k, where k is the rank of the algebra, that are the *Cartan Generators*:

$$H_i = H_i^\dagger,$$

$$[H_i, H_j] = 0. \tag{2.3.1}$$

The Cartan generators commute with every other generator and form a linear space. One can choose a basis satisfying the normalization (from (2.2.1)),

$$\text{tr}\ (H_i H_j) = \lambda \delta_{ij},$$

for $i, j = 1$ to $N - 1$. For the group SU(N) $\lambda = \frac{1}{2}$. The states of the representation D can be written as

$$H_i|\mu, x, D\rangle = \mu_i|\mu, x, D\rangle, \tag{2.3.2}$$

where μ_i are the *weights*. The number of weights the fundamental representation is equal to the number of vectors on the fundamental representation, but their dimension is the dimension of the rank. For example on SU(3), one has 3 orthogonal vectors ($v_1 = (1, 0, 0)$, for instance) on the fundamental representation and 3 weights ($\mu_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$, for instance).

The contraction of a fundamental and an anti-fundamental field form as singlet is $\bar{N} \otimes N = (N^2 - 1) \oplus 1$.

2.4 The Adjoint Representation

The adjoint representation of the algebra is given by the structure constants, which are always real,

$$[X_a, X_b] = i f_{bcd} X_c. \tag{2.4.1}$$

$$[X_a, [X_b, X_c]] = i f_{bcd}[X_a, X_d] = -f_{bcd} f_{ade} X_e.$$

If there are N generators, then we find a $N \times N$ matrix representation in the adjoint representation. From the Jacobi identity,

$$[X_a[X_b, X_c]] + [X_b[X_c, X_a]] + [X_c[X_a, X_b]] = 0, \tag{2.4.2}$$

one has similar relation for the structure constants,

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0.$$

Defining a set of matrices T_a as

$$[T_a]_{bc} \equiv -i f_{abc},$$

it is possible to recover (2.1.2):

$$[T_a, T_b] = i f_{bcd} T_c.$$

The states of the adjoint representation correspond to the generators $|X_a\rangle$. A convenient scalar product is:

$$\langle X_a | X_b \rangle = \lambda^{-1} \text{ tr } (X_a^\dagger X_b).$$

The action of a generator in a state is:

$$\begin{aligned} X_a |X_b\rangle &= |X_c\rangle \langle X_c | X_a | X_b \rangle \\ &= |X_c\rangle [T_a]_{cb} \\ &= i f_{abc} |X_c\rangle \\ &= |i f_{abc} X_c\rangle \\ &= |[X_a, X_b]\rangle. \end{aligned}$$

The Killing Form

The *Killing form* is the scalar product of the algebra, defined in terms of the adjoint representation. Applying it to the generators themselves, it gives the metric $g_{\alpha\beta}$ of the Cartan matrices. The anti-symmetrical structure constants $f_{\alpha\beta\gamma} = f_{\alpha\beta}^{\gamma'} g_{\gamma'\gamma}$ have the Killing metric given by

$$g_{\gamma'\gamma} = f_{p\gamma'} f_{\gamma}^{p} \tag{2.4.3}$$
$$= \text{tr } T_{\gamma'}^{adj} T_{\gamma}^{adj}, \tag{2.4.4}$$

recalling $(T_{\alpha}^{adj})_{\beta}^{\gamma} = f_{\beta\alpha}^{\gamma}$, the trace is independent of the choice of basis.

Application on Fields

A good way of seeing the direct application of this theory on fields is, for example, the covariant derivative acting on a field in the adjoint representation:

$$\begin{aligned} (D_\mu \phi)_a &= \partial_\mu \phi_a - igA_\mu^b (t_G^a)_{bc} \phi^c, \\ &= \partial_\mu \phi_a + g f_{abc} A_\mu^b \phi^c, \end{aligned}$$

and the vector field transformation is

$$A_\mu^a \to A_\mu^a + \frac{1}{g}(D_\mu)^a.$$

2.5 The Roots

The roots are weights (states) of the adjoint representation, in the same way they were defined to the Cartan generators, (2.3.2), where from (2.3.1), we see that

$$H_i|H_j\rangle = |[H_i, H_j]\rangle = 0.$$

Therefore, all states in the adjoint representation with zero weight vectors are Cartan generators (and they are orthonormal). The other states have non-zero weight vectors α,

$$H_i|E_\alpha\rangle = \alpha_i E_\alpha. \tag{2.5.1}$$

The non-zero roots uniquely specify the corresponding states, E_α, and they are the non-hermitian raising/lowering operators:

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \tag{2.5.2}$$

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger, \tag{2.5.3}$$
$$E_\alpha^\dagger = E_{-\alpha}, \tag{2.5.4}$$

$$[E_\alpha, E_{-\alpha}] = -\alpha H, \tag{2.5.5}$$

where we can set the normalization (in the same fashion as (2.4.3)) as

$$\langle E_\alpha | E_\beta \rangle = \lambda^{-1} \text{ Tr } (E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta}.$$

From (2.5.5), for any weight μ of a representation D, setting $H = E_3$, one has

$$E_3 |\mu, x, D\rangle = \frac{\alpha\mu}{|\alpha|^2} |\mu, x, D\rangle, \tag{2.5.6}$$

which are always integers or half-integers and it is the origin of the *Master Formula*. From this equation, we can see that all roots are non-degenerated,

$$2\frac{\alpha.\mu}{\alpha^2} = -p + q, \tag{2.5.7}$$

where p is the number of times the operator E_α may raise the state and q, the number of times the operator $E_{-\alpha}$ may lower it.

Roots α

The roots can be directly calculated from the the weights of the Cartan generators by $\pm\alpha^{ij} = \mu^i \pm \mu^j$.

Positive Roots

When labeling roots in either negative or the positive, one can set the whole raising/lowering algebra. It is convention, for instance one can set for SU(N) the positive root to be the first non-vanishing entry when it is positive.

One can define an ordering of roots in the way that if $\mu > \nu$ than $\mu - \nu$ is positive, and from this finde the *highest weight* of the irrep. In the adjoint representation, positive roots correspond to raising operators and negative to lowering operators.

Simple Roots $\vec{\alpha}$

Some of the roots can be built out of others, and *simple roots* are the positive roots that cannot be written as a sum of other positive roots. A positive root is called a simple root if it raises weights by a minimal amount. Every positive root can be written as a positive sum of simple roots. If a weight is annihilated by the generator of all the simple roots, it is the *highest weight*, ν, of an irreducible representation. From the geometry of the simple roots, it is possible to construct the whole algebra,

- If $\vec{\alpha}$ and $\vec{\beta}$ are simple roots, then $\vec{\alpha} - \vec{\beta}$ is not a root (the difference of two roots is not a root).

- The angles between roots are $\frac{\pi}{2} \leq \theta < \pi$.

- The simple roots are linear independent and complete.

Fundamental Weights $\vec{q}$

Every algebra has k (rank of the algebra) *fundamental weights* that are a basis orthogonal to the simple roots α, and they can be constructed from the master formula, (2.5.7), with a metric g_{ij}, (2.4.4), to be defined,

$$2\frac{q_i^I g^{ij} \alpha_j^J}{\alpha_j^J g^{ij} \alpha_j^J} = \delta^{IJ}, \tag{2.5.8}$$

If we use of the Killing metric defined on (**??**), for SU(N), the relation becomes

$$2\frac{q^j \alpha^k}{|\alpha^k|^2} = \delta^{jk}. \tag{2.5.9}$$

All irreps can be written in terms of the fundamental weight and the *highest weight* as

$$\nu^{HW} = \sum_{i=1}^{k} a_i q^i = a_1 q^1 + a_2 q^2 + ... + a_N q^k, \tag{2.5.10}$$

where a_i are the *Dynkin coefficients*, $a_i = q_i - p_i$.

The Master Formula

We have already seen the role of the master formula on equations (2.5.6) and (2.5.7), now let us derive it properly. Supposing that the highest state is j, there is some non-negative integer p such that $(E^+)^p|\mu, x, D\rangle \neq 0$, with weight $\mu + p\alpha$, and which $(E^+)^{p+1}|\mu, x, D\rangle = 0$. The former is the *highest state* of the algebra.The value of E_3 is then

$$\frac{\alpha(\mu + p\alpha)}{\alpha^2} = \frac{\alpha\mu}{\alpha^2} + p = j. \tag{2.5.11}$$

On another hand, there is a non-negative integer q such that $(E^-)^q|\mu, x, D\rangle \neq 0$, with weight $\mu - q\alpha$, being the lowest state, with $(E^-)^{q+1}|\mu, x, D\rangle = 0$. The value of E_3 is then

$$\frac{\alpha.(\mu - q\alpha)}{\alpha^2} = \frac{\alpha.\mu}{\alpha^2} - q = -j. \tag{2.5.12}$$

Adding (9.2.2) and (9.3.4) one has the formulation of the *master formula*, as in (2.5.7),

$$\frac{\alpha.\mu}{\alpha^2} = -\frac{1}{2}(p - q).$$

Subtracting (9.2.2) from (9.3.4) one have the important relation

$$p + q = 2j. \tag{2.5.13}$$

Construction of Irreps of SU(3) from the Highest Weight ν [3]

It is possible to construct all representations of an irrep of any Lie algebra represented by its highest weight. One needs only to know a basis for the fundamental weights and their orthogonal simple roots of the group and to make use of the theory of lowering and raising operators. Let us show an example of a irrep of SU(3) with **highest weight** $\nu = q^1 + 2q^2$, where q^1, q^2 are the fundamental weight of SU(3).

Recalling from table 3.4, the simple roots and the fundamental weights of SU(3), the highest weight of the irrep we are going to construct is

$$\nu = q^1 + 2q^2 = (\frac{3}{2}, -\frac{1}{2\sqrt{3}}).$$

[3]Exercise proposed by Prof. Nieuwenhuizen[**?**].

From the master formula, (2.5.7), for each of the two simple roots (where we consider them normalized $|\alpha_1|^2 = 1 = |\alpha_2|^2$),

$$2\alpha_i\nu = \frac{1}{2}q,$$

where $p = 0$ since this is the highest weight. Calculating for the both simple roots,

$$\alpha_1\nu = q_1 = 1,$$

$$\alpha_2\nu = q_2 = 2.$$

The first vectors are then

$$|\nu\rangle, |\nu - \alpha_1\rangle, |\nu - \alpha_2\rangle, |\nu - 2\alpha_2\rangle.$$

We now lower, in the same fashion, $|\nu - \alpha_1\rangle$, with $E_{-\alpha_2}$, and $|\nu - \alpha_2\rangle$, $|\nu - 2\alpha_2\rangle$ with $E_{-\alpha_1}$,

$$2\alpha_2(\nu - \alpha_1) = q_{12} = 3,$$

giving 3 more vectors, $|\nu - \alpha_1 - \alpha_2\rangle, |\nu - \alpha_1 - 2\alpha_2\rangle, |\nu - \alpha_1 - 3\alpha_2\rangle$.

$$2\alpha_1(\nu - \alpha_2) = q_{21} = 2,$$

giving 2 more vectors, but only one new, $|\nu - \alpha_1 - 2\alpha_2\rangle$.

$$2\alpha_1(\nu - 2\alpha_2) = q_{211} = 2,$$

giving two more new vectors, $|\nu - 2\alpha_1 - 2\alpha_2\rangle, \nu - 3\alpha_1 - 2\alpha_2\rangle$.

The weights will sum up 15, which is exact the dimension of this irrep on SU(3).

2.6 The Cartan Matrix and Dynkin Diagrams

The *Cartan matrices* represent directly the proprieties of the algebra of each Lie family and are constructed from the master formula, (2.5.7), multiplying all simple roots α^i among themselves,

$$A^{ij} = 2\frac{\alpha^i\alpha^j}{|\alpha^i|^2}. \tag{2.6.1}$$

The off-diagonal elements can only be 0, -1, -2 and -3. If all roots have the same length, A is symmetric, and if $A^{ij} \neq 0$ then $A^{ji} \neq 0$. The rows of the Cartan matrix are the Dynkin coefficients (labels) of the simple root, and are directly used on the constructed of the algebra.

The *Dynkin diagram* is a diagram of the algebra of the groups in terms of angles and size of the roots. Multiplying the master formula by itself, and using the *Schwartz inequality*[4], we can define the angles between the product of two simple roots as

$$\cos\,\theta_{12} = \frac{\alpha_1\alpha_2}{|\alpha_1^2||\alpha_1^1|} = \frac{1}{4}\sqrt{n_1 n_2},$$

with the limited possibilities on $n_1 n_2 < 4$. The conditions for the two roots are then

1. $n_1 = n_2 = 0$, $\theta = \frac{\pi}{2}$, the two roots are orthogonal and there is no restriction of length. $A_{ij} = 0$, the roots are not connected.

2. $n_1 = n_2 = 1$, $\theta = \frac{\pi}{3}$, and $|\alpha_1| = |\alpha_2|$. $A_{ij} = -1$, the roots have the same length.

3. $n_1 = 2, n_2 = 1$, $\theta = \frac{\pi}{4}$, $|\alpha_2| = \sqrt{2}|\alpha_1|$. $A_{ij} = -2$, the roots have value 2 and 1.

4. $n_1 = 3, n_2 = 1$, $\theta = \frac{\pi}{6}$, $|\alpha_2| = \sqrt{3}|\alpha_1|$. $A_{ij} = -3$, the highest root has value 3.

Summarizing, the rules to construct the Dynkin diagram for some Lie algebra are the following:

1. For every simple root, one writes a circle.

2. Connect the circles by the number of lines given by A_{ij} of (2.6.1). Two circle are joined with one line if $\theta = \frac{\pi}{3}$, two lines if $\theta = \frac{\pi}{4}$, and three lines if $\theta = \frac{\pi}{6}$.

3. For a semi-simple algebra the diagram will have disjoint pieces, for example, $SO(4) \simeq SU(2) \times SU(2)$, which is not simple, is giving by two disconnected circles.

4. When the length are unequal, one can either write an arrow pointing to the root of smaller length, or write all small roots as a black dot.

From the Dynkin diagrams it is possible to check if two groups are locally isomorphic and the sequence for compact and simple Lie groups is

[4]The Cauchy-Schwarz inequality states that for all vectors x and y of an inner product space, $|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$.

Groups	Dynkin Diagram
SO(3) ≃ SU(2)≃USp(2)	∘
SO(4) ≃SU(2)×SU(2)	∘ ∘
SO(5)≃USp(4)	∘ = ∘
SO(6)≃ SU(4)	∘ − ∘ − ∘

Table 2.4: Local isomorphism among the Cartan families.

A *regular* subalgebra is obtained by deleting points from the Dynkin diagram. For example, $SU(2) \oplus SU(4) \subset SU(6)$ and its six dimensions give a regular representation of the regular subalgebra of SU(2) (1,2) and SU(4) (4,1). The *non-regular* subalgebra $SU(3) \oplus SU(2)$ gives an irrep (3,2).

There is a symmetric invariant bilinear form for the adjoint representation

$$SU(N) \subset SO(N^2 - 1). \tag{2.6.2}$$

2.7 Casimir Operators

If the Lie algebra is semi-simple, then the metric (2.4.4) has determinant non-zero ($\det g_{\alpha\beta} \neq 0$). In this case, an irrep R has the *Casimir operator* defined as

$$\tilde{C}_2(R) = g^{\alpha\beta} T^R_\alpha T^R_\beta, \tag{2.7.1}$$

where T^R are the chosen generators and the metric has the compactness condition

$$g_{\alpha\beta} = f^q_{\alpha\beta} f^p_{pq} \leq 0,$$

and the commutation relation of the Casimir operator to all other generators is zero,

$$[\tilde{C}_2(R), T^R_a] = 0.$$

Proof.

$$\begin{aligned}
[g^{\alpha\beta} T_\alpha T_\beta, T_\gamma] &= g^{\alpha\beta} T_\alpha [T_\beta, T_\gamma] + g^{\alpha\beta} [T_\alpha, T_\gamma] T_\beta \\
&= f^{\alpha\delta}_\gamma (T^R_\alpha T^k_\delta + T^k_\delta T^R_\alpha) \\
&= 0.
\end{aligned}$$

□

Using the Schur's lemma on (2.7.1), it is clear that since $\tilde{C}_2(R)$ commutes to all other generators, it must be proportional to the identity,

$$\tilde{C}_2(R) = C_2(R)I,$$

where $C_2(R)$ is the *Quadratic Casimir Invariant* of each irrep, which is an *invariant* of the algebra. It has a meaning only for representations, not as an element of the Lie algebra, since the product (2.7.1) are not defined for the algebra itself, but only for the representations.

For compact groups, the Killing form is just the Kronecker delta, for example on SU(2), the Casimir invariant is then simply the sum of the square of the generators L_x, L_y, L_z of the algebra, i.e., the Casimir invariant is given by $L^2 = L_x^2 + L_y^2 + L_z^2$. The Casimir eigenvalue in a irrep is just $L^2 = l(l+1)$. For example for the adjoint irrep,

$$f^{acd} f^{bcd} = C_2(G)\delta^{ab},$$

a symmetric invariant two-indice tensor $\delta_{\alpha\beta} = \delta_{\beta\alpha}$ is unique up rescaling $(T_\gamma^{adj})_\alpha^\beta$, therefore it is possible to write tr $T_\alpha^R T_\beta^R = \delta_{\alpha\beta}^R = \delta_{\alpha\beta} T(R)$.

To compute explicitly T(R), one starts with any Lie algebra tr $T_\alpha T_\beta = T(R) g_{\alpha\beta}$, and the Casimir operators can be found from knowing the dimension of the irrep R and the group G,

$$C_2(R) \times \dim R = \dim G \times T(R), \text{ or,} \tag{2.7.2}$$

$$\sum_a T_a(R)^2 = C_2(R) \times 1_{d(R)\times d(R)}. \tag{2.7.3}$$

For the fundamental representation of SU(N), $C_2(R) = \frac{N^2-1}{2N}$. For the adjoint representation $C_2(G) = C(G) = N$. For the spinorial representation of SO(2N), $C_2(G) = 2^{N-4}$. The number of generators required to give a complete set of these invariants is equal to the rank.

Harish-Chandra Homomorphism

The center of the algebra of the semi-simple Lie algebra is a polynomial algebra. The degrees of the generated algebra are the degree of the fundamental invariants. The number of invariants on each family is shown on table 2.5.

Example: The fundamental irrep 3 of SU(3)

For the irrep 3 of SU(3), one has the generators given by

$$T_a(3) = \frac{\lambda_a}{2},$$

A_n	$I_2, I_3, ..., I_{n+1}$
B_n	$I_2, I_4, ..., I_{2n}$
C_n	$I_2, I_4, ..., I_{2n}$
D_n	$I_2, I_4, ..., I_{2n-2}$

Table 2.5: Order of the independent invariants for the four Lie families.

and the dimension of the group is $N^2 - 1 = 8$ resulting

$$\text{tr} \sum_a (\frac{\lambda_a}{2})^2 = \sum_a \frac{1}{2}\delta^{aa} = \frac{1}{2} \times 8 \rightarrow 3C_2(3),$$

the quadratic Casimir invariant is then

$$C_2(3) = \frac{4}{3}.$$

Example: The adjoint irrep 8 of SU(3)

Now one has

$$T_a(8) = i f_{abc},$$

where

$$-\sum_a f_{abc} f_{dbc} = C_2(8)\delta_{ad}.$$

The quadratic Casimir invariant is given by

$$8C_2(8) = f_{abc}^2 = 6(1 + \frac{6}{4} + \frac{23}{24}) = 24,$$

$$C_2(8) = 3.$$

2.8 *Weyl Group

For every root m there is a state with $-m$: large algebras have reflection symmetries and the group generated by those reflections are the *Weyl group*. This group maps weights to weights.

2.9 *Compact and Non-Compact Generators

The number of compact generators less the number of non-compact is the rank of the Lie Algebra, which is the maximum number of commuting generators. On table 2.6 the Cartan series are separated in terms of their compact and non-compact generators, which is given by the following algebra.

$$[C, C] = C,$$

$$[C, NC] = NC,$$

$$[NC, NC] = C.$$

A_n, SU(N+1)	N compact T_i where $[T_i, T_j] = \epsilon_{ijk} T_k$.
A_n, SL(N,C)	SL(N,R), non-compact: all generators of SU(N) times i. SU(p,q), non-compact, $\sum_{i=1}^{p} (x^i)^* x^i = \sum_{j=p+1}^{N} (x^j)^* x^j$.
B_n, SO(2N+1,C)	SO(2N+1,R), real, compact. SO(p,q,R), non-compact, $\sum_{i=1}^{p} (x_i)^2 = \sum_{j=p+1}^{N} (x^j)^2$.
D_n, SO(2N,C)	SO(2N,R), real, compact. SO(p,q), p+1=2N, non-compact. Examples: Anti-deSitter superalgebra, SO(4,1), SO(2,N)*, deSitter superalgebra SO(3,2), SO(N).

Table 2.6: The Lie groups in terms of the compact and non-compact generators.

Example of non-compact groups are SU(p,q) (which preserves the form $x^\dagger 1_{p,q} y$) and SL(N,R), which is the group of $N \times N$ real matrices with unitary determinant. One can go from compact groups to non-compact versions by judiciously multiplying some of the generators by i (or equivalently, letting some generators become pure imaginary). The general procedure for associating a non-compact algebra with a compact one is first to find the maximal subalgebra and then multiply the remaining non-compact generators by i.

2.10 *Exceptional Lie Groups

The process of constructing the algebra for the five Cartan exceptional groups G_2, F_4, E_6, E_7, E_8, consists basically on the following method:

1. Choose an easy subgroup H of G and a simple representation of this H. This should be the *Maximal Regular subgroup*, i.e. an algebra which has the same rank thus we can write the Cartan generators of the group as a linear combination of the Cartan generators of the subgroup, and there is no large subalgebra containing it except the group itself.

2. Decompose the representation R into irreps R_i of H, and how it acts in R_i. An irrep of an algebra becomes a representation of the subalgebra when its is embedded by a homomorphism that preserves the commutation relations.

3. Construct the Lie algebra of G starting with the Lie algebra of H and R_i of H.

For example, mainly from (2.6.2) one can find the following: E_6 has a maximal regular subalgebra in $SU(6) \otimes SU(2)$, E_7 has a maximal subalgebra in $SU(8)$, and and E_8 has a maximal regular subalgebra in SO(16). The lowest dimensional irrep of E_8 is the adjoint irrep with 24-dimensional generators, which can be used on the above process.

As a remark, before these exceptional families, the first groups are actually locally isomorphic to the four Lie families, $E_5 \simeq SO(10)$, $E_4 \simeq SU(3) \times SU(2)$, and $E_3 \simeq SU(2) \times SU(2)$.

Chapter 3

SU(N), the A_n series

SU(N) is the group of $N \times N$ unitary matrices, $U^\dagger U = 1$ with det $U = 1$. The rank of SU(N) is $N-1$ and the number of generators is N^2-1. The traceless constraint $tr(\alpha_\alpha T_\alpha) = 0$ is what gives the determinant a unitary value.

Proof. Any element of the group can be represented as

$$U(\alpha) \to e^{i\alpha T}$$

and can be diagonalized by

$$U\alpha T U^{-1} = D,$$

Now, taking the determinant,

$$\begin{aligned} \det U(\alpha) &= \det\left(e^{iD}\right) \\ &= e^{i \text{ tr } D} \\ &= e^{i \text{ tr } \alpha T}. \\ &= 1. \end{aligned}$$

□

3.1 The Defining Representation

SU(N) has N objects $\phi^i, i = 1, ..., N$ that transform under $\phi^i \to \phi'^i = U^i_j \phi^j$. The complex conjugate transforms as $\phi^{*i} \to \phi'^{*i} = (U^i_j)^* \phi^{*j} = (U^\dagger)^j_i \phi^{*j}$.

There are higher representations, for example, the tensor ϕ^{ij}_k, transform as they were equal to multiplication of the vectors, $\phi^i\phi^j\phi_k$,

$$\phi^{ij}_k \to U^i_l U^j_m (U^\dagger)^n_k \phi^{lm}_n.$$

The trace is always a singlet and it can be separated by setting the upper index equals to the lower index, $\phi^{ij}_j \to U^i_l \phi^{lm}_m$, and subtracting it from the original tensor. In this way, the tensor ϕ^{ij}_k can be decomposed into sets containing $\frac{1}{2}N^2(N+1)-N$ symmetric traceless and $\frac{1}{2}N^2(N-1)-N$ anti-symmetric traceless components. More examples can be seen on table 3.1.

Element	Dimension	Description	Dimension on SU(5)
ϕ^i	N	Defining Irrep	5
ϕ^{ij}	N^2	$N \otimes N$ Defining	25
ϕ^i_j	N^2-1	$N \otimes \bar{N} - U(1)$, Adjoint irrep	24
ϕ^{ij}	$\frac{N(N+1)}{2}$	Symmetric	15
ϕ^{ij}	$\frac{N(N-1)}{2}$	Anti-symmetric	10
ϕ^{ij}_k	N^3	$N \otimes N \otimes N$, Defining	125
ϕ^{ij}_k	$\frac{1}{2}N^2(N+1)-N$	Symmetric traceless	70
ϕ^{ij}_k	$\frac{1}{2}N^2(N-1)-N$	Anti-symmetric traceless	45
ϕ^{ik}_k	2N	Trace	10

Table 3.1: The decomposition of SU(N) into its irreps.

3.2 The Cartan Generators H

The Cartan-Weyl basis for the group SU(N) are the maximally commutative basis of $N-1$ generators, $N-1$, given can given by the following basis of matrices, which are a generalization of the Gell-Mann matrices:

$$H^I = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... \end{pmatrix}_{N\times N}$$

$$H^{II} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... \end{pmatrix}_{N\times N}$$

$$H^{III} = \frac{1}{\sqrt{24}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & ... \end{pmatrix}_{N\times N}$$

...

$$H^{N-1} = \frac{1}{\sqrt{2N(N-1)}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & ... & -(N-1) \end{pmatrix}_{N\times N}$$

3.3 The Weights μ

The weights of the defining representation, from the Cartan generators, are N-vectors with $N-1$-entries each, given by

$$\begin{aligned}
\mu^{I} &= (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{24}}, ..., \frac{1}{\sqrt{2N(N-1)}}), \\
\mu^{II} &= (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{24}}, ..., \frac{1}{\sqrt{2N(N-1)}}), \\
\mu^{III} &= (0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{24}}, ..., \frac{1}{\sqrt{2N(N-1)}}), \\
&\;\;...\\
\mu^{N} &= (0, 0, 0, ..., \frac{-(N-1)}{\sqrt{2N(N-1)}}),
\end{aligned}$$

3.4 The Roots α

The roots are the weights of the generators that are not Cartan. There are $N(N-1)$ roots on SU(N) and together with N weights, they give N^2, which is one plus the number of total generators (given by N^2-1). The roots on SU(N) are generated by the commutation between these generators,

$$[H, E_\alpha] = \alpha_i E_\alpha,$$

$$[H, E_{\alpha_{ij}}] = (\mu^i - \mu^j) E_{\alpha_{ij}},$$

$$\pm\, \alpha_{ij} = \mu^i - \mu^j j. \tag{3.4.1}$$

The Positive Roots

The positive roots are given by the first non-vanishing positive entry. There are $\frac{N(N-1)}{2}$ positive roots on SU(N).

The Simple Roots $\vec{\alpha}$

Simple roots are positive roots that cannot be constructed by others. The number of simple roots is the equal to the rank k of the algebra, where $k = N - 1$ in the case of SU(N). In resume, to find the simple roots, one calculates all possible roots by

$$\vec{\alpha}^{ij} = \mu^i - \mu^j, \tag{3.4.2}$$
$$= \mu^i - \mu^{i+1} \to \alpha^{12}, \alpha^{23}, .., \alpha^{N-1,N}, \tag{3.4.3}$$

finds the positives roots, and then check which are not the sum of others (totalizing $N - 1$ simple roots).

When associating roots and fundamental weights to the Dynkin diagrams, we see that the fundamental highest weights are the first weight of the defining representation and its complex conjugate (last weight). The symmetric product of two vectors (2-fold) gives two HW (2,0,0...), the symmetric product of three vectors (3-fold) gives (3,0,0,0..), etc. The 3-fold anti-symmetric product contains a rep that is the sum of 3 HW, (0,0,1,0...), the two-fold anti-symmetric (0,1,0,...). To illustrate them, the highest weight for some representations for SU(6) are shown on table 3.2.

3.5 The Fundamental Weights $\vec{q}$

The $k = N - 1$ fundamental weights of a Lie algebra are given by their orthogonality to the simple roots, relation that can be checked with the master formula (2.5.7),

$$2\frac{\alpha^I \vec{q}^J}{|\alpha|^2} = \delta^{IJ}. \tag{3.5.1}$$

For SU(N), one can also write the fundamental weights in a basis doing

$$q^i = \mu^i - \mu^{i+1}$$

Representation	HW on the Dynkin Diagram
Fundamental	◦ − ◦ − ◦ − ◦ 1 - 0 - 0 - 0 - 0
Anti-fundamental	◦ − ◦ − ◦ − ◦ 0 - 0 - 0 - 0 -1
2-fold Symmetric	◦ − ◦ − ◦ − ◦ 2 - 0 - 0 - 0 -0
2-fold Anti-symmetric	◦ − ◦ − ◦ − ◦ 0 - 1 - 0 - 0 - 0
Adjoint	◦ − ◦ − ◦ − ◦ 1 - 1 - 0 - 0 - 0

Table 3.2: The highest weight of all irreps of SU(6).

$$q^1 = \mu^1 - \mu^2, q^2 = \mu^2 - \mu^3, q^3 = \mu^3 - \mu^4 ....$$

resulting on

$$\begin{aligned} q^I &= (1, 0, 0, 0...0), \\ q^{II} &= (-\frac{1}{2}, \frac{\sqrt{3}}{6}, 0, 0...0), \\ q^{III} &= (0, -\frac{\sqrt{3}}{3}, \frac{\sqrt{24}}{6}, 0...0), \\ &... \end{aligned}$$

3.6 The Killing Metric

The Killing metric for SU(N) is

$$\begin{aligned} g_{ij} &= \text{tr } H_i H_j, \\ &= \frac{1}{2}\delta_{ij}. \end{aligned}$$

3.7 The Cartan Matrix

The Cartan matrix, calculated from the equation 2.6.1, from last chapter,

$$A^{ij} = 2\frac{\alpha^i \alpha^j}{|\alpha^i|^2}.$$

has the form following form on SU(N):

$$A = \begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 & 0 \\ 0 & -1 & 2 & ... & -1 & 0 & 0 \\ 0 & 0 & -1 & ... & 2 & -1 & 0 \\ 0 & 0 & 0 & ... & -1 & 2 & -1 \\ 0 & 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}.$$

3.8 SU(2)

In the *special unitary subgroup of two dimensions*, any element can be written as

$$g = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $|a^2| + |b^2| = 1$. The elements of the group are represented by $e^{\lambda_a T_a}$, with T_a antihermitian and given by the *Pauli matrices*,

$$T_a = \frac{1}{2}\sigma_a.$$

Since in SU(2) the structure constant ϵ^{ij} carries only two indices, it suffices to consider only tensors with upper indices, symmetrized. As a consequence, we are going to see that SU(2) only has real and pseudo real representations.

The Defining Representation

The Cartan generator is given by the maximal hermitian commutating basis, composed only of H, since the rank is 1. Recalling the traditional algebra from Quantum Mechanics, for the vectors that spam the fundamental defining representation, (1,0) and (0,1), and making $H = J_3$, we have

$$\begin{aligned} [J_3, J_\pm] &= \pm J_\pm, \\ H = J_3 &= \frac{1}{2}\sigma_3. \end{aligned}$$

The Weights μ

The eigenvalue of H are the weights

- $H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mu_1^I = \frac{1}{2}$.

- $H\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$, $\mu_1^{II} = -\frac{1}{2}$.

The Raising/ Lowering Operators

The Raising/ Lowering Operators are given by

$$E_+ = J_1 + iJ_2 = \frac{1}{2}(\sigma_1 + i\sigma_2),$$
$$E_- = J_1 - iJ_2 = \frac{1}{2}(\sigma_1 - i\sigma_2).$$

Extending these results, we see that for each non-zero pair of root vector $\pm\alpha$, there is a SU(2) subalgebra with generators

$$E^{\pm} \equiv |\alpha|^{-1} E_{\pm\alpha},$$
$$E_3 \equiv |\alpha|^{-2} \alpha H.$$

The generators, the weights and the simple root of the defining (fundamental) representation are shown on table 3.3.

Cartan generator	H	$\frac{1}{2}\sigma_3$
Raising operator	E^+	$\frac{1}{2}(\sigma_1 + i\sigma_2)$
Lowering operator	E^-	$\frac{1}{2}(\sigma_1 - i\sigma_2)$
Weight I	μ^I	$\frac{1}{2}$
Weight II	μ^{II}	$-\frac{1}{2}$
Simple Root I	$\vec{\alpha}^{12}$	1
Fundamental Weight I	$\vec{q}$	1

Table 3.3: The generators, weights, simple root and fundamental weight of the defining representation of SU(2).

3.9 SU(3)

The elements of SU(3) are given by $e^{b_a T_a}$, with b_a real and T_a traceless and antihermitian that can be constructed from the original *Gell-Mann matrices*, λ_a,

$$T_a = \frac{1}{2}\lambda_a.$$

From these matrices one has three compact generators, which are those from SU(2) plus five extra non-compact generators. The number of compact generators less the number of non-compact is the rank of the Lie Algebra, k, which is the maximum number of commuting generators. In this case $k = 3 - 1 = 2$, giving the two Cartan generators

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\lambda_3$$

$$H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{2\sqrt{3}}\lambda_8$$

The other $N^2 - 1 - k = 6$ generators of SU(3), are

$$\lambda_{1,2,3} = \begin{pmatrix} \sigma_i & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The Defining Representation

The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are the basis of the defining representation.

The Weights μ

The eigenvalues of the Cartan generators on the defining representation give the threes weights of this algebra,

$$\begin{aligned} \mu^I &= (\frac{1}{2}, \frac{1}{2\sqrt{3}}), \\ \mu^{II} &= (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), \\ \mu^{III} &= (0, -\frac{1}{\sqrt{3}}). \end{aligned}$$

The Raising/ Lowering Operators

The raising/ lowering operators of SU(3) are

$$E^I_\alpha = i(\lambda_1 + i\lambda_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^{II}_\alpha = i(\lambda_4 + i\lambda_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^{III}_\alpha = i(\lambda_6 + i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^I_{-\alpha} = i(\lambda_1 - i\lambda_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^{II}_{-\alpha} = i(\lambda_4 - i\lambda_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E^{III}_{-\alpha} = i(\lambda_6 - i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The Roots α

The roots are the weight (states) of the adjoint representation. The first generator can be written as

$$[H_1, E_{\alpha^I}] = E^I_\alpha,$$

$$[H_2, E_{\alpha^I}] = 0,$$

concluding that the roots is $\alpha^I_+ = (1,0)$, and the root of $(E^I_\alpha)^\dagger = E^I_{-\alpha}$ is just the same vector with opposite sign, $\alpha^I_- = (-1,0)$. For the second generator and its complex conjugate,

$$[H_1, E_{\alpha^{II}}] = \frac{1}{2} E^{II}_\alpha,$$

$$[H_2, E_{\alpha^{II}}] = \frac{\sqrt{3}}{2} E_{\alpha}^{II},$$

thus the root are $\pm\alpha^{II} = \pm(\frac{1}{2}, \frac{\sqrt{3}}{2})$. For the third generator, the roots are $\pm\alpha^{III} = \pm(-\frac{1}{2}. \frac{\sqrt{3}}{2})$.

The Simple Roots $\vec{\alpha}$

The simple roots of SU(3) are those that cannot be constructed by summing any two positive roots,

$$\vec{\alpha}^I = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$
$$\vec{\alpha}^{II} = (\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

The Fundamental Weights $\vec{q}$

The two fundamental weights on SU(3) represent the 3 and $\bar{3}$ irreps. By applying (3.5.1) one gets $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$.

The Cartan Matrix

From (2.6.1), the product of the two simple roots are given by

$$\begin{aligned} \cos\theta_{11} = 2\vec{\alpha}_1\vec{\alpha}_1 &= -2, \\ \cos\theta_{12} = 2\vec{\alpha}_1\vec{\alpha}_2 &= -1, \\ \cos\theta_{22} = 2\vec{\alpha}_2\vec{\alpha}_2 &= -2. \end{aligned}$$

$$A_{SU(3)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The group SU(3) has an important role on *phenomenology of elementary particles*. For instances one can represents mesons (quark and anti-quark) as $3 \otimes \bar{3} = 8 \oplus 1$, $\psi_a \otimes \psi^b = (\psi_a\bar{\psi}^b - \frac{1}{3}\delta_a^b\psi_c\bar{\psi}^c) + \frac{1}{3}\delta_a^b\psi_c\bar{\psi}^c$, and baryons (3 quarks) as $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$, $\psi_a \otimes \psi_b = \frac{1}{2}(\psi_{(a}\psi_{b)} + \psi_{[a}\psi_{b]})$.

Example[2]: Working out in another Representation of SU(3)

Let us suppose another (natural) way of choosing H_j and $E_{\pm\alpha}$, given by

[2]Exercise proposed by Prof. Nieuwenhuizen.

Cartan generators	H_3, H_8	$\frac{1}{2}\lambda_3,\ \frac{1}{2}\lambda_8$
Raising operator	$E_{+\alpha_i}, i = 1,2,4,5,6,7$	$\frac{1}{2}(\lambda_i + \lambda_j)$
Lowering operator	$E_{-\alpha_i}, i = 1,2,4,5,6,7$	$\frac{1}{2}(\lambda_i - \lambda_j)$
Weight I	μ^I	$(\frac{1}{2}, \frac{1}{2\sqrt{3}})$
Weight II	μ^{II}	$(-\frac{1}{2}, \frac{1}{2\sqrt{3}})$
Weight III	μ^{III}	$(0, \frac{-1}{\sqrt{3}})$
Simple Root I	$\vec{\alpha}^I$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
Simple Root II	$\vec{\alpha}^{II}$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
Fundamental Weight I	q^I	$(\frac{1}{2}, \frac{1}{2\sqrt{3}})$
Fundamental Weight II	q^{II}	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$

Table 3.4: The generators, weights, simple roots and fundamental weights of the defining representation of SU(3).

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$E_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

with $E_{-\alpha} = (E_\alpha)^\dagger$. The weights of the defining representation are then giving by

$$\begin{aligned} \mu^I &= (\frac{1}{2}, 0), \\ \mu^{II} &= (-\frac{1}{2}, \frac{1}{2}), \\ \mu^{III} &= (0, -\frac{1}{2}). \end{aligned}$$

and the six roots are given by

$$\begin{aligned} \alpha &= (1, -\frac{1}{2}), \\ \beta &= (\frac{1}{2}, \frac{1}{2}), \\ \gamma &= (-\frac{1}{2}, 1), \end{aligned}$$

and their negative values. The positive roots are given by the roots with first entry non-negative,

$$\begin{aligned}\alpha &= (1, -\frac{1}{2}),\\ \beta &= (\frac{1}{2}, \frac{1}{2}),\\ -\gamma &= (\frac{1}{2}, -1).\end{aligned}$$

The quantity of simple roots are given by the rank of the algebra, in this case, $k = 2$, from (3.4.3), one has

$$\begin{aligned}\alpha^{12} &= \alpha - \beta = (\frac{1}{2}, 1),\\ \alpha^{23} &= \beta - (-\gamma) = (0, -\frac{3}{2}).\end{aligned}$$

The fundamental highest weights q^j is given by using the Killing metric $g_{ij} = \text{ tr } (H_i H_j)$,

$$2\frac{\alpha_i^I g^{ij} q_j^J}{\alpha_i^I g^{ij} \alpha_i^I} = 2\frac{\alpha^J \mu^I}{\alpha^{J2}} = \delta^{IJ},$$

giving

$$q^I = (\frac{1}{2}, 0) \text{ and } q^{II} = (0, -\frac{1}{2}).$$

Chapter 4

SO(2N), the D_n series

SO(2N) is the group of matrices O that are orthogonal: $O^T O = 1$ and have det $O = 1$. The group is generated by the imaginary antisymmetric $2N \times 2N$ matrices, which only $2N^2 - N$ are independent (which is exactly the number of generators, table 2.3). The explicitly difference to the group SU(N) is that there the group was represented by both upper and lower indices, however, on SO(2N), this distinction of indices has no meaning. The rank of SO(2N) is $N = n$.

4.1 The Defining Representation

The defining representation is the $2N$ vectors $\vec{v} = \{v^i, i = 1..., 2N\}$ which transforms as $v^i \to v'^i = O^{ij} v^j$. Possible representations for the tensors are the $(2N)^2$ objects given by T^{ij}, the $(2N)^3$ given by $T^{ijk} \to T'^{ijk} = O^{il} O^{jm} O^{kn} T^{lmn}$, etc. It is possible to decompose any tensor into symmetrical and antisymmetrical subsets, for example for T^{ij}, one has $\frac{1}{2}N(N+1)$ and $\frac{1}{2}N(N-1)$, respectively:

$$S \to S^{ij} = \frac{1}{2}(T^{ij} + T^{ji}) \tag{4.1.1}$$

$$A \to A^{ij} = \frac{1}{2}(T^{ij} - T^{ji}). \tag{4.1.2}$$

Giving the symmetrical tensor T^{ij} and considering its trace as $T = \delta^{ij} T^{ij}$

then

$$\begin{aligned} T \to \delta^{ij} T'^{ij} &= \delta^{il} O^{ij} O^{jm} T^{lm}, \\ &= (O^T)^{li} \delta^{ij} O^{jm} T^{lm}, \\ &= (O^T)^{lj} O^{jm} T^{lm}, \\ &= \delta^{lm} T^{lm}, \\ &= T, \end{aligned}$$

which means that the trace transforms to itself (singlet). Therefore, it is possible to subtract it from the original tensor, forming the traceless $Q^{ij} = T^{ij} - \frac{1}{N}\delta^{ij} T$ which are $\frac{1}{2}N(N+1) - 1$ elements transforming among themselves.

To summarize it, given two vectors v and w, their product can be decompose into a symmetric traceless, a trace and an anti-symmetric tensor:

$$N \otimes N = [\frac{1}{2}N(N+1) - 1] \oplus 1 \oplus \frac{1}{2}N(N-1). \tag{4.1.3}$$

For example, for SO(3), $3 \otimes 3 = 5 \oplus 1 \oplus 3$.

4.2 The Cartan Generators H

For the group SO(2N), the N Cartan generators can be represented generically by the $2N \times 2N$ following matrices

$$[H_m]_{jk} = -i(\delta_{j,2m-1}\delta_{k,2m} - \delta_{k,2m-1}\delta_{j,2m}), \tag{4.2.1}$$

$$H_1 = -i \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2N\times 2N},$$

$$H_2 = -i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2N\times 2N},$$

...

$$H_N = -i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}_{2N\times 2N} .$$

4.3 The Weights μ

The 2N weights of the defining representation, for the previous Cartan generators, are $\pm$ the unit vector e^k with components $[e^k]_m = \delta_{km}$,

$$\begin{aligned} \mu^1 &= (1,0,0,...,0)_N, \\ \mu^2 &= (-1,0,0,...,0), \\ \mu^3 &= (0,1,0,...,0), \\ \mu^4 &= (0,-1,0,...,0), \\ &... \\ \mu^{2N-1} &= (0,0,0,...,1), \\ \mu^{2N} &= (0,0,0,...,-1). \end{aligned}$$

4.4 The Raising and Lowering Operators $E^{\pm}$

In the group SO(2N), the raising and lowering operators are given by a collection of $2N^2 - 2N$ operators represented by

$$E_\alpha = E_{IJ}^{\eta\eta'}, \qquad (4.4.1)$$

where $\eta = \pm 1$, $\eta' = \pm 1$ (giving four possibilities), and $IJ = 1, ..., N$. These operators can be explicitly written as

$$E_{12}^{\eta\eta'} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & i\eta' & 0 \\ 0 & 0 & i\eta & -\eta\eta' & 0 \\ -1 & -i\eta' & 0 & 0 & 0 \\ -i\eta & \eta\eta' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2N\times 2N} .$$

$$E_{N-1,N}^{\eta\eta'} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i\eta' \\ 0 & 0 & 0 & i\eta & -\eta\eta' \\ 0 & -1 & -i\eta' & 0 & 0 \\ 0 & -i\eta & \eta\eta' & 0 & 0 \end{pmatrix}_{2N\times 2N} .$$

4.5 The Roots α

The $2N^2 - 2N$ roots are given by the N-size positive roots such as $\pm e^j \pm e^k$, for $k \neq j$,

$$\begin{aligned}
\alpha^1 &= (1,1,0,...,0)_N, \\
\alpha^2 &= (-1,1,0,...,0), \\
\alpha^3 &= (0,1,1,0,...,0), \\
\alpha^4 &= (0,-1,1,0,...,0), \\
\alpha^5 &= (1,0,1,...,0), \\
\alpha^6 &= (-1,0,1,...,0), \\
...\alpha^{2N^2-2N} &= (0,0,-0,...,-1,-1),
\end{aligned}$$

Positive Roots

The positive roots on SO(2N) are defined by the roots with first positive non-vanishing entry, $\alpha = (0,0,..,0,+1,0...)$, i.e. $e^j \pm e^k$ for $j < k$.

Simple Roots $\vec{\alpha}$

The N simple roots are given by $N-1$ vectors $e^j - e^{j+1}, j = 1...N-1$ and one $e^{N-1} + e^N$,

$$\begin{aligned}
\vec{\alpha}^1 &= (1,-1,0,...,0), \\
\vec{\alpha}^2 &= (0,-1,-1,...,0), \\
&... \\
\vec{\alpha}^{N-1} &= (0,0,..,1,-1), \\
\vec{\alpha}^N &= (0,0,...,1,1).
\end{aligned}$$

4.6 The Fundamental Weights $\vec{q}$

The N fundamental weights of SO(2N) are

$$\begin{aligned}
\vec{q}^1 &= (1, 0, 0, ..., 0), \\
\vec{q}^2 &= (1, 1, 0, ..., 0), \\
\vec{q}^3 &= (1, 1, 1, ..., 0), \\
&... \\
\vec{q}^{N-1} &= \frac{1}{2}(1, 1, 1, ..., -1), \\
\vec{q}^N &= \frac{1}{2}(1, 1, 1, ..., 1),
\end{aligned}$$

where the last two are the *spinor representation* and the *conjugate spinor representation*. The complex representation is characterized by charge conjugation, so all weights change sign.

4.7 The Cartan Matrix

The Cartan Matrix in SO(2N) is given by (2.6.1),

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & ... & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & ... & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & ... & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 & ... & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & ... & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & ... & -1 & 0 & 2 \end{pmatrix}.$$

Chapter 5

SO(2N+1), the B_n series

The algebra of SO(2N+1) is the same as SO(2N) with an extra dimension. The group is generated by the imaginary antisymmetric $2N \times 2N$ matrices, which only $2N^2 + N$ are independent (which is exactly the number of generators, table 2.3).

5.1 The Cartan Generators H

The defining representation is $2N + 1$-dimensional, therefore the N Cartan generators are given by the following $(2N+1) \times (2N+1)$ matrices, where one just adds zeros on the last collum and row of the previous SO(2N) Cartan generators. The ranking of SO(2N+1) is N.

$$H_1 = -i \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2N+1 \times 2N+1} ,$$

$$H_2 = -i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2N+1 \times 2N+1} ,$$

...

$$H_N = -i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2N+1 \times 2N+1},$$

5.2 The Weights μ

The $2N+1$ weights of SO(2N+1) are the N-sized vectors from SO(2N) plus an extra vector with zero entries, all them given by

$$\begin{aligned} \mu^1 &= (1,0,0,...,0)_N, \\ \mu^2 &= (-1,0,0,...,0), \\ &... \\ \mu^{2N} &= (0,0,...,0,-1), \\ \mu^{2N+1} &= (0,0,...,0,0). \end{aligned}$$

5.3 The Raising and Lowering Operators $E^{\pm}$

The raising and lowering operators are the same as in SO(2N), $E_\alpha = E_{IJ}^{\eta\eta'}$, equation (4.4.1), where $\eta = \pm 1$, $\eta' = \pm 1$, $I, J = 1, ..., N$, plus N more operators respecting

$$[E_I^\eta, E_J^{\eta'}] = -E_{IJ}^{\eta\eta'}.$$

These new $\frac{1}{2}(2N+1)2N$ operators E_I^η are given by

$$E_1^\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & i\eta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & i\eta & 0 & 0 & 0 \end{pmatrix}_{2N+1 \times 2N+1}.$$

...

$$E_N^\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & i\eta \\ 0 & 0 & -1 & i\eta & 0 \end{pmatrix}_{2N+1 \times 2N+1}.$$

5.4 The Roots α

The roots of SO(2N+1) are the same of SO(2N) plus $2N$ vectors for the new raising/lowering operators, given by $\pm e^j$,

$$\alpha^{IJ\eta\eta'} = (..., \pm 1, ...0, \pm 1, 0...)_N,$$
$$\text{and}$$
$$\alpha^{I\eta} = (..., \eta^J, ...).$$

Positive Roots

Again, the positive roots on SO(2N+1) are the same as on SO(2N), defined by the left entry, plus a new positive root e^j,

$$\alpha^1 i = (0, 0, 1, ..., \pm 1, 0, ..., 0).$$

Simple Roots $\vec{\alpha}$

The N simple roots of SO(2N+1) are given by $e^j - e^{j+1}$ for $j = 1, ..., n-1$ and e^N (the last one is different from the SO(2N) case, since $e^{N-1} + e^N$ is not simple simple here).

$$\begin{aligned}
\vec{\alpha}^1 &= (1, -1, 0, ..., 0), \\
\vec{\alpha}^2 &= (0, 1, -1, ..., 0), \\
&... \\
\vec{\alpha}^{N-1} &= (0, 0, .., 1, -1), \\
\vec{\alpha}^N &= (0, 0, 0, ..., 1).
\end{aligned}$$

5.5 The Fundamental Weights $\vec{q}$

The N fundamental weights of SO(2N+1) are

$$\begin{aligned}
\vec{q}^1 &= (1, 0, 0, ..., 0)_N, \\
\vec{q}^2 &= (1, 1, 0, ..., 0), \\
&... \\
\vec{q}^{N-1} &= (1, 1, 1..., 1, 0), \\
\vec{q}^N &= \frac{1}{2}(1, 1, 1..., 1),
\end{aligned}$$

where the last is the self-conjugated spinor representation.

5.6 The Cartan Matrix

The Cartan matrices is different from SO(2N) and identifies this family, B_n,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & ... & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & ... & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & ... & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 & ... & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & ... & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}.$$

Chapter 6

Spinor Representations

Spinor representations are irreps of the orthogonal group, SO(2N+2) and SO(2N+1). One can write these generators in a basis formed of gamma-matrices γ^i, respecting the Clifford Algebra, $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. The hermitian generators of the rotational group will be then the matrices formed by $M_{ij} = \frac{1}{4i}[\gamma_i, \gamma_j] = \sigma_{ij}$.

A way of visualizing it is that, for groups SO(2N+2), one has the spinor (S) irrep, $\sigma_{ij} = \{\gamma_{ij}, ..., -i\gamma_{ij}\}$ and the conjugate $(\bar{S})$ irrep, $\bar{\sigma}_{ij} = \{\gamma_{ij}, ..., i\gamma_{ij}\}$.An example for SO(4=2.1+1) is constructed on table 6.1. For groups SO(2N+1), the spinor is its conjugate, so the representation is always real.

	Spinor (S)	Complex $(\bar{S})$
Representation	$\lvert\frac{1}{2}\rangle \otimes \lvert\frac{1}{2}\rangle$	$\lvert\frac{1}{2}\rangle \otimes \lvert -\frac{1}{2}\rangle$
Cartan Generators	$H_1^S = \sigma_3 \otimes \sigma_3$	$H_2^C = -\sigma_3 \otimes \sigma_3$
Ladder Generators	$E_1^S = \sigma_1 \otimes 1$	$E_2^C = \sigma_2 \otimes 1$

Table 6.1: Example of spinor representation for SO(4), an euclidian space of dimension 4. This representation is pseudo-real, which is not a surprise since $SO(4) \simeq SU(2) \otimes SU(2)$ and $SU(2)$ is pseudo-real.

6.1 The Dirac Group

The *Dirac matrices* (composed from the Pauli matrices) form a group. For instance, let us consider an euclidian four-dimensional space. We can write the Dirac matrices as $\gamma^4 = i\gamma^0$. The Dirac group is then of order $2^{N+1} = 32$ and the elements of this group are G $= +I, -I, \pm\gamma^\mu, \gamma^{\mu\nu}, \gamma^{\mu\nu\rho}, \gamma^{1234}$. There are 17 classes, given by the orthogonality relation, (1.2.5),

$$32 = \sum^{17}(\text{ dim } R^i)^2 = 16 \times 1^2 + 4^2,$$

where 16 are one-dimensional irreps that do not satisfy the *Clifford Algebra* given by (6.1.1), i.e. only the four-dimensional irrep does so. Other examples are shown on table (6.2).

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \eta^{\mu\nu}. \tag{6.1.1}$$

6.2 Spinor Irreps on SO(2N+1)

In the $2N + 1$ dimension of the defining representation that we have just constructed last chapter, we had the fundamental weights for $j = 1, ..., N-1$ given by

$$\vec{q}^j = \sum_{k=1}^{j} e^k, \tag{6.2.1}$$

and Nth-fundamental weight was

$$\vec{q} = \frac{1}{2}\sum_{k=1}^{N} e^k. \tag{6.2.2}$$

The last is the spinor irrep and by *Weyl reflections* in the roots e^j, it gives the set of weights

$$\frac{1}{2}(\pm e^1, \pm e^2, ..., \pm e^N),$$

where all of them are uniquely equivalent by rotation to the highest weight of some 2^N-dimensional representation. This representation is a tensor product of N 2-dimensional spaces, where any arbitrary matrix can be built as a

	Euclidian, d=2
Elements	$\{+I, -I, \pm\gamma_1, \pm\gamma_2, \pm\gamma_3\gamma_2\}$
Classes	5
Order $[G]$	8
Orthogonality	$8 = \sum^5 R_i^2 = 1 + 1 + 1 + 2^2$
Clifford Algebra	1
C_+	I
C_-	σ_2
Reality	real
	Euclidian, d= 3
Elements	$\{+I, -I, \pm\gamma_1, \pm\gamma_2, \pm\gamma_3, \pm\gamma_1\gamma_2, \pm\gamma_3\gamma_2, \pm\gamma_3\gamma_1, -\gamma_{123}, +\gamma_{123}\}$
Classes	10
Order $[G]$	16
Orthogonality	$16 = \sum^{10} R_i^2 = 16 \times 1 + 2^2 + 2^2$
Clifford Algebra	2
C_+	No solution
C_-	σ_2
Reality	pseudo-real
	Euclidian and Minkowski, d=4
Elements	$\{+I, -I, \pm\gamma_\mu, \pm\gamma_\mu\gamma_\nu, \pm\gamma_\mu\gamma_{nu}\gamma_\rho, \pm\gamma_1\gamma_2\gamma_3\gamma_4\}$
Classes	17
Order $[G]$	32
Orthogonality	$32 = \sum^{10} R_i^2 = 16 \times 1 + 4^2$
Clifford Algebra	1
Reality	Euclidian: pseudo-real, Minkowskian: real (Majorana)

Table 6.2: The Dirac group for dimensions 2, 3 and 4 (euclidian and minkowskian).

tensor product of Pauli matrices, i.e. the set of states which forms the spinors irrep is a $2N$-dimensional spaces given by

$$|\pm\frac{1}{2}\rangle_1 \otimes |\pm\frac{1}{2}\rangle_2 \;...\; \otimes |\pm\frac{1}{2}\rangle_N.$$

In this notation, the Cartan generators are

$$H^i = \frac{1}{2}\sigma_3^i,$$

generalized as the following hermitian generators

$$M_{2k-1,2N+1} = \frac{1}{2}\sigma_3^1...\sigma_3^{k-1}\sigma_1^k, \tag{6.2.3}$$
$$M_{2k,2N+1} = \frac{1}{2}\sigma_3^1...\sigma_3^{k-1}\sigma_2^k. \tag{6.2.4}$$

All other generators can be constructed from the relation

$$M_{ab} = -i[M_{a,2N+1}, M_{b,2N+1}],$$

for $a, b \neq 2N-1$. The lowering and raising operators are clearly

$$\begin{aligned}
E^{\pm}_{e^1} &= \frac{1}{2}\sigma_{\pm}^{e^1}, \\
E^{\pm}_{e^2} &= \frac{1}{2}\otimes\sigma_3^{e^1}\otimes\sigma_{\pm}^{e^2}, \\
E^{\pm}_{e^3} &= \frac{1}{2}\otimes\sigma_3^{e^1}\otimes\sigma_3^{e^2}\otimes\sigma_{\pm}^{e^3}, \\
&... \\
E^{\pm}_{e^j} &= \frac{1}{2}\sigma_3^{e^1}\otimes...\otimes\sigma_3^{e^{j-1}}\otimes\sigma_{\pm}^{e^j}.
\end{aligned}$$

and because we can only raise the state in this representation once, $E^2_{e^j} = 0$.

The γ-matrices of the Clifford algebra are generically given by $2^N \times 2^N$ matrices

$$\begin{aligned}
\gamma_{2k-1} &= 1\otimes 1\otimes ...\otimes 1\otimes\sigma_1\otimes\sigma_3\otimes\sigma_3\otimes...\otimes\sigma_3, \\
\gamma_{2k} &= 1\otimes 1\otimes ...\otimes 1\otimes\sigma_2\otimes\sigma_3\otimes\sigma_3\otimes...\otimes\sigma_3,
\end{aligned}$$

where 1 appears $k-1$ times and σ_3 appears $N-k$ times. Explicitly they are

$$
\begin{aligned}
&\gamma_1 = \sigma_1 \otimes \sigma_3 ... \sigma_3, \\
&\gamma_2 = \sigma_2 \otimes \sigma_3 ... \sigma_3, \\
&\gamma_3 = 1 \otimes \sigma_1 \otimes \sigma_3 ... \sigma_3, \\
&\gamma_4 = 1 \otimes \sigma_2 \otimes \sigma_3 ... \sigma_3, \\
&... \\
&\gamma_{2N-1} = 1 \otimes ... 1 \otimes \sigma_1 \\
&\gamma_{2N} = 1 \otimes ... 1 \otimes \sigma_2 \\
&\gamma_{2N+1} = \sigma_3 \otimes ... \otimes \sigma_3
\end{aligned}
$$

6.3 Spinor Irreps on SO(2N+2)

For SO(2N+2), besides the fundamental weights given by (6.2.1) we have two more fundamental weights

$$
\begin{aligned}
\vec{q}^N &= \frac{1}{2}(e^1 + e^2 + ... + e^N - e^{N+1}) \to S, \\
\vec{q}^{N+1} &= \frac{1}{2}(e^1 + e^2 + ... + e^N + e^{N+1}) \to \bar{S}.
\end{aligned}
$$

In this case one has one more hermitian Cartan generator for each of two spinor irreps (spinor and complex conjugate of spinor) from the SO(2N+1) case. Therefore the generators of SO(2N+2) are the the previous generators of SO(2N+1) plus for each of the two complex spinor representation:

$$
\begin{aligned}
H_{N+1} = \frac{1}{2}\sigma_3^1 ... \sigma_3^{N+1} \to S, \\
H_{N+1} = \frac{1}{2}\sigma_3^1 ... \sigma_3^{N+1} \to \bar{S}.
\end{aligned}
$$

The generators of the group are functions of γ-matrices, in the same fashion as (6.2.4). Explicitly the $2^N \times 2^N$ matrices are

$$\begin{aligned}
\gamma_1 &= \sigma_1 \otimes \sigma_3 ... \sigma_3, \\
\gamma_2 &= \sigma_2 \otimes \sigma_3 ... \sigma_3, \\
\gamma_3 &= 1 \otimes \sigma_1 \otimes \sigma_3 ... \sigma_3, \\
\gamma_4 &= 1 \otimes \sigma_2 \otimes \sigma_3 ... \sigma_3, \\
&... \\
\gamma_{2N-1} &= 1 \otimes ... 1 \otimes \sigma_1 \\
\gamma_{2N} &= 1 \otimes ... 1 \otimes \sigma_2
\end{aligned}$$

Notice that there is a non-trivial matrix that anticommutes to all others, he γ^5 from field theory, in a generalized dimension:

$$\gamma^{FIVE} = (-1)^N \gamma_1 \gamma_2 ... \gamma_{2N} \quad (6.3.1)$$

$$= \sigma_3 \otimes \sigma_3 \otimes ... \otimes \sigma_3, \text{ (N-times)}. \quad (6.3.2)$$

The projection into left-handed and right-handed spinors cut the number of components into half, thus the 2-irrep spinor of SO(2N) has dimension 2^{N-1}. For example, SO(10) has $2^{N-1} = 2^4 = 16$ dimensions.

6.4 Reality of the Spinor Irrep

We have already talked about reality of representations for finite groups, on section 1.3. Here, again, to test the reality conditions of the spinor representation, M_{ij}, one needs to find a matrix C that makes a similarity transformation $M'_{ij} = C M_{ij} C^{-1}$, for $1 \leq i < j \leq 2N, 2N+1$. The matrix C has the form

$$C = \prod_{odd} \sigma_2 \otimes \prod_{even} \sigma_1,$$

and we call it the *charge conjugate* $C^{-1} \sigma^*_{ij} C = -\sigma_{ij}$, which means that charges $e^{i\theta_i \sigma^i}$ will be charge conjugated by this operation. We resume these proprieties on table 6.3 and the classification of reality for the groups SO(2N+2) and SO(2N+1) can be seen at table 6.4. The reality propriety of spinors can also be analyzed from (6.3.2),

$$C^{-1} \gamma^{FIVE} C = (-1)^N \gamma^{FIVE}. \quad (6.4.1)$$

$(M_{ij})^* = CM_{ij}C^{-1}$	Hermitian T_α satisfies M_{ij}
	$C^T = (-1)^{\frac{N(N+1)}{2}} C$
Real	symmetric $C = C^T$,
Pseudo-real	anti-symmetric $C = -C^T$
$(M_{ij})^* \neq CM_{ij}C^{-1}$	No M_{ij} solutions, S interchanges to $\bar{S}$
	$2n+2$, n even: complex irrep, $S, \bar{S}$.

Table 6.3: The definition for reality of spinor irrep for the orthogonal group.

SO(2 + 8k)	complex
SO(3 + 8k)	pseudoreal
SO(4 + 8k)	pseudoreal
SO(5 + 8k)	pseudoreal
SO(6 + 8k)	complex
SO(7 + 8k)	real
SO(8 + 8k)	real
SO(9 + 8k)	real
SO(10 + 8k)	complex

Table 6.4: The classification of reality of spinor irrep for the orthogonal group.

6.5 Embedding SU(N) into SO(2N)

The group SO(2N) leaves $\sum_{j=1}^{N}(x'_j x_j + y'_j y_j)$ invariant. The group U(N) consists on the subset of those transformations in SO(2N) that leaves invariant also $\sum_{j=1}^{N}(x'_j y_j - y'_j x_j)$.

The defining representation of SO(2N), a vector representation of dimension 2N, decomposes upon restriction to U(N) to $N, \bar{N}$:

$$2N \to N \oplus \bar{N}.$$

The adjoint representation of SO(2N) which has dimension $N(2N-1)$ transforms under restriction to U(N) as

$$2N \otimes_A 2N \to (N \oplus \bar{N}) \otimes_A (N \oplus \bar{N}),$$

where $\otimes_A$ is the anti-symmetric product, meaning that SO(2N) decompose on U(N) as

$$N(2N-1) \to N^2 - 1(\text{adjoint}) \oplus 1 \oplus \frac{N(N-1)}{2} \oplus \Big(\frac{N(N-1)}{2}\Big)^*.$$

The embedding of SU(N) into SU(2N) in terms of irreps D^i is shown on table 6.5.

SO(2n+2)	$D^{2n+1} = \sum_{j=0}^{n}[2j+1]$	$D^{2n} = \sum_{j=0}^{n}[2j]$
SO(4n)	$D^{2n} = \sum_{j=0}^{n}[2j]$	$D^{2n-1} = \sum_{j=0}^{n-1}[2j+1]$

Table 6.5: The embedding of SU(N) on the spinor representation of SO(2N).

Chapter 7

Sp(2N), the C_n series

The *symplectic groups*, Sp(2N), are formed by matrices M that transforms as

$$\Omega = M\Omega M^T \tag{7.0.1}$$

$$M\Omega + \Omega M^T = 0, \tag{7.0.2}$$

where

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The rank of Sp(2N) is N and it has $2N^2 + N$ generators. The subgroup USp(2N) is a simple and compact group formed by the intersection of SU(2N) and Sp(2N,C), i.e. the compact form of Sp(2N),

$$USp(2N) = SU(N) \cup Sp(2N).$$

From SU(N) we automatically get the additional condition to (7.0.2) condition $M^\dagger M = -M$. The general form of M of USp(2N) is composed by the algebra of S(N),

$$M_{ij} = A_i \otimes I + \sum_{j=1}^{3} iS_j \otimes \sigma_j.$$

If A_i is real, M is anti-symmetric, otherwise, if S_j is real, M is symmetric. The number of non-compact generators minus the number of compact generator equal to the rank of the Lie algebra, therefore we can decompose it as

$$M = (A \otimes I + S_2 \otimes i\sigma_2) + (S_1 \otimes i\sigma_1 + S_3 \otimes i\sigma_3).$$

7.1 The Cartan Generators H

The first N-1 generators are the same Cartan subalgebra of SU(N) in the first block and its complex conjugate on the second (charge conjugation, so all weights change sign),

$$H^1 = \frac{1}{2}\begin{pmatrix} 1 & & & ... & & \\ & -1 & & & & \\ & & 0 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & 0 \end{pmatrix}_{2N\times 2N},$$

$$H^2 = \frac{1}{2}\begin{pmatrix} 1 & & & ... & & & & \\ & 1 & & & & & & \\ & & -2 & & & & & \\ & & & 0 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 2 & \\ & & & & & & & 0 \end{pmatrix}_{2N\times 2N},$$

...

$$H^{N-1} = \frac{1}{\sqrt{2N(N-1)}}\begin{pmatrix} 1 & & & ... & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & -(N-1) & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & (N-1) \end{pmatrix}_{2N\times 2N},$$

The Nth Cartan generator is given by

$$H^N = \frac{1}{\sqrt{2N(N-1)}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{2N\times 2N}.$$

7.2 The Weights μ

The 2N weights of the defining representation of USp(2N) are

$$\begin{aligned}
\mu^1 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{24}}, ..., \frac{1}{\sqrt{2(N-1)N}}, \frac{1}{\sqrt{2N}}\right), \\
\mu^2 &= \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{24}}, ..., \frac{1}{\sqrt{2(N-1)N}}, \frac{1}{\sqrt{2N}}\right), \\
\mu^3 &= \left(0, -\frac{2}{2\sqrt{12}}, \frac{1}{\sqrt{24}}, ..., \frac{1}{\sqrt{2(N-1)N}}, \frac{1}{\sqrt{2N}}\right), \\
&... \\
\mu^{N-1} &= \left(0, 0, ..., 0, \frac{N-1}{\sqrt{2(N-2)(N-1)}}, ...\right), \\
\mu^N &= \left(0, 0, ..., 0, -\frac{N-1}{\sqrt{2N(N-1)}}, \frac{1}{\sqrt{2N}}\right), \\
\mu^{N+1} &= -\mu^1, \\
&... \\
\mu^{2N} &= -\mu^N.
\end{aligned}$$

7.3 The Raising and Lowering Operators $E^{\pm}$

For $j < k$, one construct the raising operators of USp(2N),

$$E_{1,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$E_{1,3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{2N\times 2N},$$

$$E_{2,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{2N\times 2N},$$

...

$$E_{1,2+N} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{2N\times 2N},$$

$$E_{1,3+N} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{2N\times 2N},$$

$$E_{2,3+N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{2N\times 2N},$$

...

The lowering operators are given by $(E_{ij})^{\dagger} = -E_{\bar{i}\bar{j}}^{-}$. The commutation relation

$$[H_I, E_\alpha] = \alpha_I E_\alpha,$$

$$[H_I, E_\alpha] = (H_{I,jj} - H_{I,kk})E_{jk},$$

shows that E_α are eigenvalues of the H matrices.

7.4 The Roots α

The roots from the raising and lowering operators are give by $\alpha_{j,k} = \mu_j - \mu_k$ and $\alpha_{j,k+N} = \mu_j + \mu_k$,

$$\pm(\mu^i - \mu^j),\ 1 \le i < j \le N,$$
$$\pm(\mu^i + \mu^j),\ 1 \le i, j \le N.$$

Positive Roots

The positive roots are defined as the first positive entry from the right

$$\mu^i - \mu^j,\ \ \text{i} < j,$$
$$\mu^i + \mu^j,\ \forall\ \text{i , j} .$$

Simple Roots

The N simple roots of USp(2N) are given by

$$
\begin{aligned}
&\alpha_1 = (1, 0, 0, ..., 0), \\
&\alpha_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, ..., 0), \\
&... \\
&\alpha_{N-1} = (0, ..., 0, -\frac{N-2}{\sqrt{2(N-2)(N-1)}}, \frac{N}{\sqrt{2N(N-1)}}), \\
&\alpha_N = (0, ..., 0, -\frac{2(N-2)}{\sqrt{2N(N-1)}}, \sqrt{\frac{2}{N}}).
\end{aligned}
$$

7.5 The Fundamental Weights q

The N fundamental weights are

$$
\begin{aligned}
&q^1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, ..., \frac{1}{\sqrt{2N}}), \\
&... \\
&q^{N-1} = (0, 0, ..., 0, \sqrt{\frac{N-1}{2N}}, \frac{N-1}{\sqrt{2N}}), \\
&q^N = (0, 0, ..., 0, \sqrt{\frac{N}{2}}).
\end{aligned}
$$

7.6 The Cartan Matrix

The Cartan matrices identifies the family C_n,

$$
A = \begin{pmatrix}
2 & -1 & 0 & 0 & ... & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & ... & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & ... & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & ... & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & ... & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & ... & 0 & -2 & 2
\end{pmatrix}.
$$

Chapter 8

Young Tableaux

On the *Young tableaux* theory, each tableau represents a specific process of *symmetrization* and *anti-symmetrization* of a tensor $v^{12\ldots n}$ in which index n can take any integer value 1 to N.

For instance, lets us recall the defining representation of SU(N), composed of N-dimensional vectors v^{μ} (it represents the usual vectors $|\mu\rangle$, $\mu = 1, ..., N$). The tensor product is $u^{\mu} \otimes v^{\nu}$, with N^2 components, is

$$\delta(u^{\mu}v^{\nu}) \;\; = \;\; T^{\nu}_{\mu'}u^{\mu'}v^{\nu} + u^{\mu}T^{\mu}_{\nu'}v^{\nu'}.$$

This tensor product forms irreps, i.e the. $u^{\mu} \otimes v^{\nu}$ is reducible into irreps, and in the case that the representations are identical, the product space can be separated into two parts, a symmetric and antisymmetric part,

$$u^{\mu} \otimes v^{\nu} \;\; = \;\; \frac{1}{2}(u^{\mu}v^{\nu} + u^{\mu}v^{\nu}) \oplus \frac{1}{2}(u^{\mu}v^{\nu} - u^{\nu}v^{\mu}). \tag{8.0.1}$$

The decomposition of the fundamental representation specifies how a subgroup H is embedded in the general group G, and since all representations may be built up as products of the fundamental representation, once we know the fundamental representation decomposition, we know all the representation decompositions.

The Young tableaux are then the box-tensors with N indices, where first one symmetrizes indices in each row, and second one antisymmetrizes all indices in the collums. For example, we can construct the Young diagrams, from (8.0.1), for the tensorial product $u^{\mu}v^{\nu} \rightarrow \boxed{\mu} \otimes \boxed{\nu}$ as

$$\boxed{\mu} \otimes \boxed{\nu} = \boxed{\mu}\boxed{\nu} + \begin{array}{c}\boxed{\mu}\\ \boxed{\nu}\end{array} = t^{\mu\nu} + t^{\nu\mu}$$

Another example, for $N > 2$, we have the tensor $t^{\mu\rho\nu} = v^{\mu} \otimes u^{\rho} \otimes h^{\nu}$, and one writes

$$t^{\mu\nu\rho} \longrightarrow \left(\boxed{\mu} \otimes \boxed{\nu} \right) \otimes \boxed{\rho} =$$

$$\left(\begin{array}{|c|c|}\hline \mu & \nu \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \end{array} \right) \otimes \boxed{\rho} =$$

$$\begin{array}{|c|c|c|}\hline \mu & \nu & \rho \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline \mu & \rho \\ \hline \nu \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline \mu & \nu \\ \hline \rho \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \rho \\ \hline \end{array}$$

where, for example,

$$t^{\mu\rho\nu} \longrightarrow \begin{array}{|c|c|}\hline \nu & \rho \\ \hline \mu \\ \cline{1-1} \end{array} \rightarrow (t^{\mu\nu\rho} + t^{\mu\rho\nu}) - (t^{\nu\mu\rho} + t^{\nu\rho\mu}).$$

A third example, for $N > 4$, one can have a tensor of the form $t^{\mu\nu\rho\gamma\eta}$, which will be generalized as

$$\boxed{\mu} \otimes \boxed{\nu} \otimes \boxed{\rho} \otimes \boxed{\gamma} \otimes \boxed{\eta} = \begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline \end{array} \oplus ... \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array}$$

Direct Products of two Irreps

The formal calculation of the direct product of two (or more) irreps is given by the following rules:

1. On the second tensor in the tensorial multiplication, add to all boxes of the first row a indices, add to all boxes of the second row b indices, etc. For example, $\begin{array}{|c|c|}\hline a & a \\ \hline b \\ \cline{1-1} \end{array}$.

2. The process of multiplication is made adding one box of index a each time to the others boxes, in all allowed ways, until ceasing these boxes. Then adding one box of index b each time to all, etc. Never put more than one a or b in the same column. Align the number of a's $\leq$ number of b's (at any position, number of a above and right must be equal or smaller than of b, which must be equal or smaller than c, etc). For example, $\begin{array}{|c|c|c|}\hline a & a & b \\ \hline \end{array}$ is not allowed.

3. If two tableaux of the same shape are produced, they are counted as different only if the labels are different.

4. Cancel columns with N boxes since they are the trivial representation.

5. Check the dimension of the products to the dimension of the initial tensors.

Example $v^{\mu\nu} \otimes u^{\rho}$

$$\ytableaushort{bb} \otimes \ytableaushort{a} = \ytableaushort{bba} + \ytableaushort{bb,a}$$

Example $u^{\rho} \otimes v^{\mu\nu}$

The last one of this product is forbidden:

$$\ytableaushort{b} \otimes \ytableaushort{aa} = \ytableaushort{baa} + \ytableaushort{ba,a} + \ytableaushort{b,a,a}$$

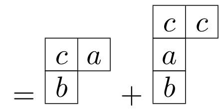
Example $v^{\mu} \otimes u^{\nu}_{\rho}$

The first three of this product are forbidden:

$$\ytableaushort{c} \otimes \ytableaushort{a,b} = \left(\ytableaushort{ca} + \ytableaushort{c,a}\right) \otimes \ytableaushort{b}$$
$$= \ytableaushort{cab} + \ytableaushort{cb,a} + \ytableaushort{cb,a} + \ytableaushort{c,a,b}$$

Example $v^{\mu\nu} \otimes u^{\rho}_{\lambda}$

$$\ytableaushort{cc} \otimes \ytableaushort{a,b} = \left(\ytableaushort{ca} + \ytableaushort{cc,a}\right) \otimes \ytableaushort{b}$$



Group	Symmetric tensors, $\square\square$	Dimension
SU(N)	$v^{\mu\nu}$	$\frac{1}{2}N(N+1)$
SO(N)	$v^{\mu\nu} - \frac{1}{N}\delta^{\mu\nu}$	$\frac{1}{2}N(N+1)-1$
USp(2N)	$v^{\mu\nu}$	$\frac{1}{2}(2N)(2N+1)$
Group	Anti-symmetric tensors, $\begin{array}{c}\square\\\square\end{array}$	Dimension
SU(N)	$v^{\mu\nu}$	$\frac{1}{2}N(N-1)$
SO(N)	$v^{\mu\nu}$	$\frac{1}{2}N(N-1)$
USp(2N)	$v^{\mu\nu} - \Omega^{\mu\nu}(\Omega^{\rho\sigma}v_{\rho\sigma})$	$\frac{1}{2}(2N)(2N-1)$

Table 8.1: Symmetric and anti-symmetric representations of the Lie families.

8.1 Invariant Tensors

A invariant tensor is a scalar (with indices) in an irreps of G, such that if one transforms the tensor, according to the indices, the invariant tensor does not change, also called *singlet*. The trace is usually a singlet, as we had proved at section 4.1.

Invariant Tensor	Group
$\delta^{\mu\nu}$	SO(2N),SU(N)
$\Omega^{\mu\nu}$	Sp(2N)
$(\gamma^\mu)^\alpha_\beta$	SO(2N+1)
$(\sigma^\mu)^{A\dot{B}}$	SO(2N)
$e^{\mu_i - \mu_n}$	SO(N), Sp(N), SU(N)

Table 8.2: The invariant tensors for the Lie families.

For SO(2N), one can write the symmetric tensor $t^{\mu\nu}$ as $u^{(\mu}v^{\nu)} - \frac{1}{N}\delta^{\mu\nu}uv$, where it was subtracted the trace, with dimension $\frac{1}{2}N(N+1) - 1$. For the antisymmetric part, the trace is not subtracted and the dimension is $\frac{1}{2}N(N-1)$. For the symplectic group Sp(2N), one writes an antisymmetric vector as $u^{\mu}v^{\nu} - u^{\nu}u^{\mu} - \frac{2}{N}\Omega^{\mu\nu}uv$, which dimension is $\frac{1}{2}(2N)(2N-1)$.

For example, the multiplication in SU(5) of $\bar{5}$ and 10 is given by the tensor $t_k^{ij} = \phi_k \eta^{ij}$. We separate out the trace $\phi_k \eta^{kj}$ which transforms as 5 and we get $\bar{5} \otimes 10 = 5 \oplus 45$.

Another example the multiplication of 10 and 10, T^{ij}_{mnh}, taking the trace and separate it out, T^{ij}_{mij} is $\bar{5}$, and the traceless, T^{ij}_{mnj} is $\bar{4}5$. Therefore $10 \otimes 10 = \bar{5} \oplus \bar{4}5 \oplus \bar{5}5$.

A third example is for Usp(4), where one can decompose the tensor product of two vectors as

$$u^{\mu}v^{\nu} = \frac{u^{\mu}v^{\nu} + u^{\nu}u^{\mu}}{2} + [u^{\mu}v^{\nu} - u^{\nu}v^{\mu} - \frac{1}{4}\Omega^{\nu\mu}(\Omega_{\rho\sigma}u^{\rho}v^{\sigma})],$$

where the last part is the trace. The tableaux representation is

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} + \ydiagram{1,1} + \bullet.$$

8.2 Dimensions of Irreps of SU(N)

The general formula to calculate the dimension of irreps of SU(N) is given by **Dimension = Factors/Hooks**, where factors are the terms inside the boxes and hook is the product of number of boxes in each hook. The dimensions of the simplest representations are

$$\dim(\ydiagram{1}) = \text{N, fundamental representation of } v^{\mu},$$

$$\dim(\ydiagram{1} \otimes \ydiagram{1}) = \ydiagram{2} + \ydiagram{1,1} = N^2$$

$$\dim(\ydiagram{2}) = \tfrac{1}{2}N(N+1), \text{ symmetric representation,}$$

$$\dim(\ydiagram{1,1}) = \tfrac{1}{2}N(N-1), \text{ anti-symmetric representation,}$$

$$\dim(\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1}) = \ydiagram{3} + 2\,\ydiagram{2,1} + \ydiagram{1,1,1} = N^3,$$

$$\dim\,(\ydiagram{3}) = \begin{pmatrix} N+2 \\ 3 \end{pmatrix} = \frac{N(N+1)(N+2)}{3!},$$

$$\dim\left(\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array}\right) = \begin{pmatrix} N \\ 3 \end{pmatrix} = \frac{N(N-1)(N-2)}{3!},$$

$$\dim\left(\begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{|c}{}\\\cline{1-1}\end{array}\right) = N^3 - \tfrac{N(N+1)(N+2)}{3!} - \tfrac{N(N-1)(N-2)}{3!} = \tfrac{2N^3}{3} - \tfrac{2N}{3}.$$

For example, for **SU(3)**, one has dim $(\Box) = 3$, represented by v^μ, and dim $(\Box \times \Box) = 3^2$, represented by $u^{\mu\nu}$. For **SU(6)**, dim $(\Box) = 6$, represented by w^μ.

The adjoint representation is the one with $N-1$ boxes on first column and only one box on second collumn. The complex conjugate of a irrep can be found by replacing the j-column element by the $N-j$, and reading from right. For example, for SU(3), one has

$$(\Box)^* = \left(\begin{array}{|c|}\hline \\\hline \\\hline\end{array}\right).$$

For SU(4), one has

- $(\Box)^* = \begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array},$

- $\left(\begin{array}{|c|}\hline \\\hline \\\hline\end{array}\right)^* = \begin{array}{|c|}\hline \\\hline \\\hline\end{array},$

- $\left(\begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{|c}{}\\\cline{1-1}\end{array}\right)^* = \begin{array}{|c|c|}\hline &\\\hline &\\\hline &\multicolumn{1}{|c}{}\\\cline{1-1}\end{array},$

- $\left(\begin{array}{|c|c|}\hline &\\\hline &\\\hline\end{array}\right)^* = \begin{array}{|c|c|}\hline &\\\hline &\\\hline\end{array},$

The Tensor Representation for SU(N)

The relation between the Young tableaux, the tensors of the group SU(N) and the fundamental weight is given by

$$\boxed{\mu} \longrightarrow v^{\mu}, \mu = 1, ..., N,\ \vec{q}^{1} = \mu^{1}$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \longrightarrow v^{\mu\nu} = -v^{\nu\mu},\ \vec{q}^{2} = \mu^{1} + \mu^{2}.$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \longrightarrow v^{\mu\nu\rho},\ \vec{q}^{3} = \mu^{1} + \mu^{2} + \mu^{3}.$$

$$...$$

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \longrightarrow \vec{q}^{N-1} = \mu^{1} + ... + \mu^{N-1} = (\square)^{*}$$

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \longrightarrow \text{For N boxes, this is a scalar.}$$

When it comes to the complex representation, the highest weight of $\square^{*}$ is the lowest weight of $\square$,

$$\begin{aligned} (\square)^{*} &= -\mu^{N}, \\ &= \mu^{1} + ... + \mu^{N-1}, \\ &= q^{N-1}. \end{aligned}$$

If the highest weight of two representations are v_i, v_2, the highest weight of the product of these representations are $v_1 + v_2$.

8.3 Dimensions of Irreps of SO(2N)

To count the dimension of a irrep on SO(2N), we write the Young tableau and fill it with the values of r_i, λ_i (number of box in the i row, counting from N-1, N-2,...), λ_i (number of boxes forming a column on the i row), and R_i (sum of r_i and λ_i), as shown on table 8.3.

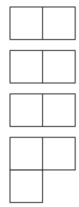
r_i (rows)		λ_i (columns)	$R_i = r_i + \lambda_i$
N-1		+2	N-1 +2
N-2		+2	N-2 +2
...		+2	... +2
1		+1	2
0		0	0

Table 8.3: Calculation of the dimension of irreps on SO(2N).

The dimension of the tensors (without spinors) is then calculated by multiplying all possible sums of R_i, multiplying all possible differences between them and dividing by all possible sums of r_i times and their differences,

$$dimension = \frac{\prod \text{ Sums } (R_i + R_j)}{\prod \text{ Sums } (r_i + r_j)} \frac{\prod \text{ Differences } (R_i - R_j)}{\prod \text{ Differences } (r_i - r_j)}.$$

For spinors one just needs to make $R_i = r_i + \lambda_i + \frac{1}{2}$.

Example: SO(6)

Let us calculate an irrep on SO(6). For instance, the multiplication of two vectors can be written as

$$v^\mu \otimes t^\nu = v^\mu t^\nu + v^\nu t^\mu - \frac{2}{N} g^{\mu\nu} vt. \tag{8.3.1}$$

Considering first of all the symmetrical part, one has the table 8.4.

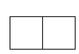
r_i (rows)		λ_i (columns)	$R_i = r_i + \lambda_i$
2		2	4
1		0	1
0		0	0

Table 8.4: Example of counting dimensions for a symmetric irrep on SO(6).

$$dimension = \frac{5.4.1 \times 3.4.1}{3.2.1 \times 1 \times 2.1} = 20.$$

r_i (rows)		λ_i (columns)	$R_i = r_i + \lambda_i$
2	□	1	3
1	□	1	2
0		0	0

Table 8.5: Example of counting dimensions for an anti-symmetric irrep on SO(6).

We then consider the anti-symmetric part, as shown on table 8.5.

$$dimension = \frac{5.3.1 \times 1.3.2}{3.2.1 \times 1 \times 2.1} = 15.$$

The dimension of the tensorial product given by (8.3.1) is clearly 20+15+1, where this last part is the trace. Now, let us consider the the dimension of spinor representation on SO(6), placing a dot inside the tableaux, as on table 8.6.

r_i (rows)		λ_i (columns)	$R_i = r_i + \lambda_i$
2	⊡	1	$3\ \frac{1}{2}$
1		0	$1\frac{1}{2}$
0		0	$\frac{1}{2}$

Table 8.6: Example of counting dimensions for a irrep on SO(6) with spinor representation.

$$dimension = \frac{5.4.2 \times 2.3.1}{3.2.1 \times 1 \times 2.1} = 20.$$

8.4 Dimensions of Irreps of SO(2N+1)

Already including spinors ($R_i = r_i + \lambda_i + \frac{1}{2}$) one can count the dimensions of the irreps with the following rules of table 8.7.

The dimension for the tensor are then again the multiplication of sums and differences of R_i over the multiplication of sum and differences of r_i,

$$dimension = \frac{\prod \text{ Sums } (R_i + R_j)}{\prod \text{ Sums } (r_i + r_j)} \frac{\prod \text{ Differences } (R_i - R_j)}{\prod \text{ Differences } (r_i - r_j)}$$

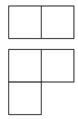
r_2 (rows)		$\lambda_i + \frac{1}{2}$ (columns)	$R_i = r_i + \lambda_i$
N+ $\frac{1}{2}$		+2	N+$\frac{1}{2}$ +2
...		+1	... 1
$\frac{1}{2}$		0	$\frac{1}{2}$

Table 8.7: Calculation of the dimension of irreps on SO(2N+1).

Example: SO(5)

First of all, considering a representation of a symmetric tensor v^μ without spinor, one has the the table 8.8.

r_i (rows)		λ_i (columns)	$R_i = r_i + \lambda_i$
$\frac{3}{2}$		1	$\frac{5}{2}$
$\frac{1}{2}$		0	$\frac{1}{2}$

Table 8.8: Example of counting dimensions for a irrep on SO(5).

The dimension is obviously 5,

$$dimension = \frac{5/2.1/2 \times 3.2}{3/2.1/1 \times 2.1} = 5.$$

Now let us count the spinor representation, writing the tensor as

$$v^{\mu\nu} = 16\Big(v^{\mu\nu} - \frac{1}{5}(\gamma^{\mu\nu})^\alpha_\beta(\gamma^\mu)^\beta\Big), \tag{8.4.1}$$

the dimension is given by table 8.9.

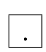
r_i (rows)		λ_i (columns)	$R_i = r_i + \lambda_i$
$\frac{3}{2}$		1 $\frac{1}{2}$	3
$\frac{1}{2}$		$\frac{1}{2}$	1

Table 8.9: Example of counting dimensions for a spinor irrep on SO(5).

$$dim = \frac{3.1.4.2}{3/2.1/2.2.1} = 16.$$

8.5 Dimensions of Irreps of Sp(2N)

The calculation of the dimension of irreps of Sp(2N) is slightly different from the previous process for the orthogonal group. Again one needs to fill the r_i, λ_i and R_i, as it is shown on table 8.10.

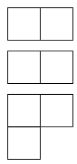
r_2 (rows)		$\lambda_i + \frac{1}{2}$ (columns)	$R_i = r_i + \lambda_i$
N		+2	N +2
...		+2	...+2
1		0	1

Table 8.10: Example of counting dimensions for a irrep on Sp(2N).

The dimension for the tensors is then given by the multiplication of all sums of R_i, all differences of R_i and (this is different) the multiplication of all R_i, all over the double factorial on $(2N-1)!!$.

$$dimension = \frac{\prod \text{ Sums } (R_i + R_j) \prod \text{ Differences } (R_i + R_j) \prod R_i}{1!3!...(2N-1)!!}$$

It is useful to check the local isomorphism of $USp(4) \simeq SO(5)$, which is clearly seen by their Young tableaux, table 8.11.

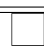
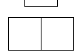
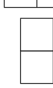
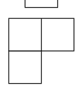
Tableau	Dimension
	4
	10
	5
	16

Table 8.11: Young tableaux from the local isomorphism $SU(5) \simeq USp(4)$.

8.6 Branching Rules

Restricted representation is a construction that forms a representation of a subgroup from a representation of the whole group. The rules for decomposing the restriction of an irreducible representation into irreducible representations of the subgroup are called *branching rules*. In the case of SU(N) groups, one can decompose it as

$$SU(N) \to SU(M) \times SU(N-M) \times U(1),$$

where the first two are represented by the diagonal matrix

$$\begin{pmatrix} SU(M) & 0 \\ 0 & SU(N-M) \end{pmatrix},$$

and the extra U(1) is embedded as

$$\begin{pmatrix} \mathrm{diag}\ \frac{1}{M} & 0 \\ 0 & \mathrm{diag}\ -\frac{1}{M-N} \end{pmatrix}.$$

The splitting in the Dynkin diagram is obtained by deleting one node, the one that connects SU(M) to SU(N-M). For example, for

$$SU(8) \to SU(4) \times SU(4) \times U(1),$$

the fundamental representation is

$$\square_N \simeq \square_M \oplus \square_{N-M}.$$

Chapter 9

The Gauge Group $SU(5)$ as a simple GUT

The idea of the *Grand Unified Theories* (GUTs) is to embed the *Standard Model* (SM) gauge groups into a large group G and try to interpret the additional resultant symmetries. Currently the most interesting candidates for G are $SU(5)$, $SO(10)$, E_6 and the semi-simple $SU(3) \times SU(3) \times SU(3)$. Since the SM group is rank 4, all G must be at least rank $N - 1 = 4$ and they also must comport complex representations. The $SU(5)$ grand unified model of Georgi and Glashow is the simplest and one of the first attempts in which the SM gauge groups $SU(3) \times SU(2) \times U(1)$ are combined into a single gauge group, $SU(5)$. The Georgi-Glashow model combines leptons and quarks into single *irreducible representations*, therefore they might have interactions that do not conserve the *baryon number*, still conserving the difference between the baryon and the lepton number (B-L). This allows the possibility of proton decay whose rate may be predicted from the dynamics of the model. Experimentally, however, the non-observed proton decay results on contradictions of this simple model, still allowing however supersymmetric extensions of it. In this paper I tried to be very explicit in the derivations of SU(5) as a simple GUT.

9.1 The Representation of the Standard Model

The current theory of the electroweak and strong interactions is based on the group $SU(3) \times SU(2) \times U_Y(1)$, henceforth called the *Standard Model of Elementary particles*. This theory states that there is a *spontaneous symmetry breaking* (SSB) at around 100 GeV, breaking $SU(2) \times U_Y(1) \to U_{EM}(1)$ via the *Higgs* mechanism. In the SM, the three generations of quarks are three identical copies of SU(3) triplet and the *right-handed* (RH) antiparticles (or *left-handed* (LH) particles) are SU(2) doublet. The remaining particles are singlet under both symmetries. For the first generation, the RH antiparticles SU(2) doublet are [1 2]

$$\bar{\psi}^{\dagger} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}, \bar{l}^{\dagger} = \begin{pmatrix} \bar{e} \\ \bar{\nu}_e \end{pmatrix}.$$

The representation of the RH antiparticle creation operators can be seen in table 9.1. For the LH particle creation one just takes the complex conjugate of RH, where for SU(2), $\bar{2} = 2$, since SU(2) is pseudo-real. The resultant operators are shown in table 9.2.

Creation Op.	Dim on SU(3)	Dim on SU(2)	Y of U(1)	Representation
$u^{\dagger}$	Triplet	Singlet	$\frac{2}{3}$	$u^{\dagger} : (3,1)_{2/3}$
$d^{\dagger}$	Triplet	Singlet	$-\frac{1}{3}$	$d^{\dagger} : (3,1)_{-1/3}$
$e^{\dagger}$	Singlet	Singlet	-1	$e^{\dagger} : (1,1)_{-1}$
$\bar{\psi}^{\dagger}$	Triplet	Doublet	$-\frac{1}{6}$	$\bar{\psi}^{\dagger} : (\bar{3},2)_{-1/6}$
$\bar{l}^{\dagger}$	Singlet	Doublet	$\frac{1}{2}$	$\bar{l}^{\dagger} : (1,2)_{1/2}$

Table 9.1: The representations of the right-handed antiparticle creation operators of the standard model, $SU(3) \times SU(2) \times U(1)$. "Dim" stands for dimension, Y is the hypercharge of the U(1) generators S.

The full $SU(3) \times SU(2) \times U(1)$ RH representation of the creation operators is then

$$u^{\dagger} \oplus d^{\dagger} \oplus e^{\dagger} \oplus \bar{\psi}^{\dagger} \oplus \bar{l}^{\dagger} = \tag{9.1.1}$$
$$(3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (\bar{3},2)_{-1/6} \oplus (1,2)_{1/2}. \tag{9.1.2}$$

The full $SU(3) \times SU(2) \times U(1)$ LH representation of the creation operators is then

$$\bar{u}^{\dagger} \oplus \bar{d}^{\dagger} \oplus \bar{e}^{\dagger} \oplus \psi^{\dagger} \oplus l^{\dagger} = \tag{9.1.3}$$
$$(\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,1)_{1} \oplus (3,2)_{1/6} \oplus (1,2)_{-1/2}. \tag{9.1.4}$$

[1] RH neutrinos weren't experimentally observed. It is possible to have only LH neutrinos without RH neutrinos if we could introduce a tiny Majorana coupling for the LH neutrinos.

[2] For the antiparticle of the electron, the positron, for convenience we write in this text $e^{+} = \bar{e}$.

Creation Op.	Dim on SU(3)	Dim on SU(2)	Y of U(1)	Representation
$\bar{u}^\dagger$	Triplet	Singlet	$-\frac{2}{3}$	$\bar{u}^\dagger : (\bar{3}, 1)_{-2/3}$
$\bar{d}^\dagger$	Triplet	Singlet	$\frac{1}{3}$	$\bar{d}^\dagger : (\bar{3}, 1)_{1/3}$
$\bar{e}^\dagger$	Singlet	Singlet	1	$\bar{e}^\dagger : (1, 1)_1$
$\psi^\dagger$	Triplet	Doublet	$\frac{1}{6}$	$\bar{\psi}^\dagger : (3, 2)_{1/6}$
$l^\dagger$	Singlet	Doublet	$-\frac{1}{2}$	$l^\dagger : (1, 2)_{-1/2}$

Table 9.2: The representations of the left-handed particle creation operators of the standard model, $SU(3) \times SU(2) \times U(1)$. "Dim" stands for dimension, Y is the hypercharge of the U(1) generators S.

The standard similar way of writing the representations of (LH) matter as representations of $SU(3) \times SU(2)$ in the SM is

$$(u, d) : (\mathbf{3}, \mathbf{2}); (\nu_e, e^-) : (\mathbf{1}, \mathbf{2}); (u^c, d^c) : (\bar{\mathbf{3}}, \mathbf{2}); (e^+) : (\mathbf{1}, \mathbf{1}). \tag{9.1.5}$$

9.2 $SU(5)$ Unification of $SU(3) \times SU(2) \times U(1)$

The breaking of $SU(5)$ into $SU(3) \times SU(2) \times U(1)$ can be done in the same fashion as the breaking of $SU(3)$ into $SU(2) \times U(1)$. The SU(5) breaking occurs when a scalar field (such as the Higgs field) transforming in its adjoint (dimension $N^2 - 1 = 5^2 - 1 = 24$) acquires a *vacuum expectation value* (VEV) proportional to the hypercharge generator,

$$S = \frac{Y}{2} = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix}. \tag{9.2.1}$$

These 24 gauge bosons are the double than the usual 12. The additional gauge bosons are called X and Y and they violate the baryon and lepton number and carry both flavor and color. As a consequence, the proton can decay into a positron and a neutral pion[3], with a lifetimes given by

$$\tau_p \sim \frac{1}{\alpha^2_{su(5)}} \frac{M_X^4}{m_p^5}.$$

[3]As we see on the last section of the text, no such decay was observed.

The $SU(5)$ is then spontaneously broken to subgroups of $SU(5)$ plus $U(1)$ (the abelian group representing the phase from (9.2.1)). This SSB can be represented as $\bar{5}\oplus 10\oplus 1$ to LH particles and $5\oplus\bar{1}0\oplus 1$ to RH antiparticles, as we will prove in the following sections.

Unbroken SU(5)

The fundamental representation of $SU(5)$, let us say the vectorial representation $|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle$, has dimension N=$5$[4]. The group is complex having an anti-fundamental representation (complex conjugate representation) $\bar{5}$. The embedding of the standard gauge groups in $SU(5)$ consists in finding a $SU(3)\times SU(2)\times U(1)$ subgroup of $SU(5)$. We first look to **5** and try to fit a 5-dimensional subset of (9.1.2) on it (and the left-handed (9.1.4) on $\bar{5}$). There is two possibilities on (9.1.2) that sum up 5 dimensions:

$$(3,1)_{-1/3}\oplus(1,2)_{1/2}, \tag{9.2.2}$$

and

$$(3,1)_{2/3}\oplus(1,2)_{1/2}.$$

The second one is not allowed because S, the generator of $U(1)$, will not be traceless[5], therefore this group can not be embed in $SU(5)$ in this case. To verify it just do $\frac{2}{3}\times 3+\frac{1}{2}\times 2\neq 0$ (not traceless), differently of $-\frac{1}{3}\times 3+\frac{1}{2}\times 2=0$ (traceless).

The first possibility, (9.2.2) is allowed and represents the embedding. The $SU(3)$ generators, T_a, act on the first indices of the fundamental rep of $SU(5)$ $|\epsilon_1\epsilon_2\epsilon_3 00\rangle$ and the $SU(2)$ generators, R_a, act on the last two $|000\epsilon_4\epsilon_5\rangle$. The $U(1)$ generator, S, obviously commutes with the other generators. The embedding of the first RH subset of fermions on $SU(5)$ is then characterized by the traceless generators

$$\begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix}_{5\times 5}, \begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix}_{5\times 5}, \begin{pmatrix} -\frac{I}{3} & 0 \\ 0 & \frac{I}{2} \end{pmatrix}_{5\times 5}.$$

In the same logic, the embedding of the LH subset of fermions on $SU(5)$ is characterized by the traceless generators

$$\begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix}_{5\times 5}, \begin{pmatrix} 0 & 0 \\ 0 & R_a \end{pmatrix}_{5\times 5}, \begin{pmatrix} \frac{I}{3} & 0 \\ 0 & -\frac{I}{2} \end{pmatrix}_{5\times 5}.$$

[4]See construction of the Lie groups on reference [?].

[5]All generators of $SU(N)$ are traceless for definition and construction.

ψ_i	$5 \to (3,1)_{-1/3} \oplus (1,2)_{1/2}$	$u^c, \bar{l}(e^c, \nu_e^c)$
ψ^i	$\bar{5} \to (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2}$	$d^c, l(e, \nu_e)$

Table 9.3: The $SU(3)\times SU(2)\times U(1)$ embedding on the fundamental representation of $SU(5)$. The index c indicates charge conjugation.

A possible representation of the LH $\bar{5}$, rewriting $|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle$[6] is[7]

$$\begin{pmatrix} d^c_{red} \\ d^c_{blue} \\ d^c_{green} \\ e \\ \nu_e \end{pmatrix}_c = \begin{pmatrix} d^c_3 \\ l_2 \end{pmatrix}_5 ,$$

and as the RH, given by **5**,

$$\begin{pmatrix} u^c_{red} \\ u^c_{blue} \\ u^c_{green} \\ \bar{e} \\ \bar{\nu}_e \end{pmatrix}_c = \begin{pmatrix} u^c_3 \\ \bar{l}_2 \end{pmatrix}_5 .$$

The next representation on $SU(5)$ we will use is the antisymmetric in two indices, **10** and its conjugate $\bar{10}$. The remaining ten-dimensional part of (9.1.2) and (9.1.4) fits respectively on $\bar{10}$ and **10**. To see how it happens, we observe that $\bar{10} = \bar{5}\otimes_A \bar{5}$ so we can multiply the 5-dimensional subsets on table 9.3 to form these representations. For instance, the LH particles (9.1.4) subset forms

$$\begin{aligned}\left[(\bar{3},1)_{\frac{1}{3}} \oplus (1,2)_{-\frac{1}{2}}\right] \quad \otimes_A \quad & \left[(\bar{3},1)_{\frac{1}{3}} \oplus (1,2)_{-\frac{1}{2}}\right] \\ = \quad & (6,1)_{\frac{2}{3}} \oplus (-3,1)_{\frac{2}{3}} \oplus (\bar{3},2)_{-\frac{1}{6}} \oplus (1,3)_{-1} \oplus (1,-2)_{-1}, \\ = \quad & (3,1)_{\frac{2}{3}} \oplus (\bar{3},2)_{-\frac{1}{6}} \oplus (1,1)_{-1}.\end{aligned}$$

where we have used $3\otimes 3 = \bar{3}\oplus 6$ and $2\otimes 2 = 1\oplus 3$ and $\otimes_A$ is the antisymmetric product (the first of each part).

Finally, all the remaining fermions transforming as a singlet (**1**) under $SU(5)$[8]. The final embedding of fermions of the standard model into the gauge group $SU(5)$ is shown on table 9.5.

[6]The index c indicates charge conjugation.
[7]Here we ignore the Cabibo type mixing.
[8]This is necessary because of the evidence for neutrino oscillations.

$10 \to (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3}$	q, u^c, e^c
$\bar{10} \to (\bar{3},2)_{-1/6} \oplus (\bar{3},1)_{2/3}$	q, d^c, e

Table 9.4: The $SU(3) \times SU(2) \times U(1)$ embedding on the anti-symmetric representation **10** of SU(5).

	$SU(5)$ Decomposition	Fermions	Similar Notation
ψ_i ψ^{ij} •	$5 \to (3,1)_{-1/3} \oplus (1,2)_{1/2}$ $\bar{10} \to (\bar{3},2)_{-1/6} \oplus (\bar{3},1)_{2/3}$ $1 \to (1,1)_0$	$u^c, \bar{l}(e^c, \nu_e^c)$ u, d^c, e ν^c	(3,1,-1/3) + (1,2,1/2) (3,1,2/3) + (3,2,-1/6)+(1,1,1)
ψ^i ψ_{ij} •	$\bar{5} \to (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2}$ $10 \to (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3}$ $1 \to (1,1)_0$	$d^c, l(e, \nu_e)$ d, u^c, e^c ν^c	($\bar{3}$,1,1/3) + (1,2,-1/2) ($\bar{3}$,1,-2/3) + (3,2,1/6)+(1,1,1)

Table 9.5: The $\bar{5} \oplus 10 \oplus 1$, LH particles, and $5 \oplus \bar{10} \oplus 1$, RH antiparticles, embedding of SM on $SU(5)$.

Breaking SU(5)

The gauge *bosons* of the model are given by the adjoint **24** of $SU(5)$, transforming as

$$24 \to (8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0 \oplus (3,2)_{-5/6} \oplus (\bar{3},2)_{5/6}, \tag{9.2.3}$$

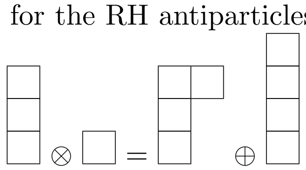
described in detail in table 9.6.

		SM GB	Add. $SU(3)$	Add. $SU(2)$	Identification
$(8,1)_0$	(8,1,0)	X	-	-	G^α_β
$(1,3)_0$	(1,3,0)	X	-	-	$W^\pm, W^0$
$(1,1)_0$	(1,1,0)	X	-	-	B
$(3,2)_{-5/6}$	(3,2,-5/6)	-	Triplet	Doublet	$A^\tau_\alpha = (X_\alpha, Y_\alpha)$
$(\bar{3},2)_{5/6}$	($\bar{3}$,2,5/6)	-	Triplet	Doublet	$A^\alpha_\tau = (X_\alpha, Y_\alpha)^T$

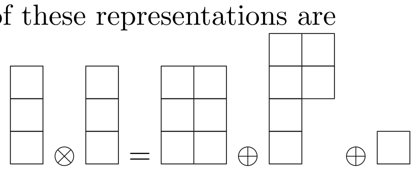
Table 9.6: The gauge bosons of SM fitting on the adjoint representation of $SU(5)$. SM stands for standard model, "Add" stands for additional, and GB for gauge boson.

As we already mentioned, the fermions have to acquire mass in $SU(5)$ by a SSB, which also must happen in the GUT theory. For making this to happen, the **product** of the representations containing the fermions and antifermions (table 9.5), must contain a component which transforms as the $SU(3) \times SU(2) \times U(1)$ Higgs field, represented by $(1,2)_{1/2}$ and $(1,2)_{-1/2}$. It is easy to understand it because the particles and antiparticles of the fermions appears in both **5,** $\bar{10}$ for RH (and **10,** $\bar{5}$ for LH).

First, for the RH antiparticles, the product of these representations are

$\bar{10} \otimes 5 =$  $= \bar{5} \oplus \bar{45}$

From (9.2.2) we see that **5** contains $(1,2)_{1/2}$, it is also contained on **45**, therefore both representations can give mass to $d, \bar{e}$. For the LH particles, the product of these representations are

$10 \otimes 10 =$  $= 50 \oplus 45 \oplus 5.$

Again **5** and **45** contains $(1,2)_{-1/2}$, giving mass to u, however **50** does not contain $(1,2)_{-1/2}$.

9.3 Anomalies

If the creation generator for all the right hand operators of spin-1/2 particles transform according to a representation generated by T_a^R, we nee to have

$$\mathrm{tr}\left[\{T_R^a, T_R^b\}T_R^c\right] = 0. \tag{9.3.1}$$

In fact it is true for all simple Lie algebra with exception of SU(N) for $N \leq 3$. Therefore, the *anomaly* in any representation of $SU(N)$ is proportional to

$$\mathcal{D}^{abc} = \mathrm{tr}\left[\{T_R^a, T_R^b\}T_R^c\right] = \frac{1}{2}A(R)d^{abc}, \tag{9.3.2}$$

where

$$2d^{abc}L^c = \{L^a, L^b\}. \tag{9.3.3}$$

$A(R)$ is independent of the generators allowing us to choose one generator and calculate it. In our case it is useful to use the generator as the charge operator

$$Q = R_3 + S, \tag{9.3.4}$$

and looking at (9.2.1), one has

$$\frac{A(\bar{5})}{A(10)} = \frac{\mathrm{tr}\ Q^3(\psi_i)}{\mathrm{tr}\ Q^3(\psi^{ii})} = -1. \tag{9.3.5}$$

It is clear now that the fermions in these representations have their anomalies canceled

$$A(\bar{5}) + A(10) = 0. \tag{9.3.6}$$

9.4 Physical Consequences of using $SU(5)$ as a GUT Theory

- The charge of the quarks can be deduced from the fact that there are three color states and from the fact that the charge operators is (9.3.4), $Q = R_3 + S = I_3 + \frac{Y}{2}$, and it must be traceless. The multiplet of the **5** representation gives

  $$Q(\nu_e) + Q(\bar{e}) + 3Q(d) = 0 \rightarrow Q(d) = -\frac{1}{3}Q(\bar{e}),$$

  which gives an answer to the charge quantization, not explained n the SM.

- The *Weinberg angle*,

$$\sin^2(\theta_W) = \frac{g'^2}{g^2 + g'^2},$$

  g, g' the coupling constants of gauge bosons in the electroweak theory, cannot be calculated on SM (it is a free parameter). In $SU(5)$ GUT, however, the Weinberg angle is accurately predicted, giving $\sin^2\theta_W \sim 0.21$.

- If a group is simple then its GUT has only one coupling constant before SSB. The three coupling constants of the SM are energy dependent and in $SU(5)$ they unit at $\sim 10^{15}$ GeV. However, a supersymmetric $SU(5)$ is needed to get an exact unification in a single point. Remember that in the SM the strong, weak and electromagnetic fine structure constants are not related in any fundamental way.

- No proton decay was observed, which is a contradiction to its lifetime estimated in $SU(5)$, (9.2.2). In a supersymmetric SU(5) the proton lifetime is longer, being apparently experimentally consistent.

- Finally, it is actually clear that the $SU(5)$ might be incomplete when one considers the fact that neutrinos were observed to carry small masses and there might exist extra RH Majorana neutrinos. As we just learned, it is not possible to introduce RH neutrinos trivially in this simple model. One solution is going to the next (complex) gauge group $SO(10)$, where the spinor representation can accommodate sixteen LH fields, or even to E_6, which motivates string theories.

Chapter 10

Geometrical Proprieties of Groups and Other Nice Features

10.1 Covering Groups

In defining representations of continuous groups, we require the matrix elements of the representation to be continuous functions on the group manifold. Among the continuous functions which are multi-valued there are possibility of multiply valued representations. A representation of a group G will be called an m-valued representation if m-different operations $D_1(R), .., D_m(R)$ are associated with each elements of the group and all these operations must be retained if the group is continuous. For all possible closed curves of the manifold, looking to the values of some function along these curves, if on returning to the initial point we find m-different values of the function, we say that the function is m-valued. A group is *simple connected* if every continuous function on the group is single valued, and it is *m-connected* if there are m closed curves which cannot be deformed into one another, i.e. m-valued continuous functions can exist, some of the irreps of the manifold are m-valued.

For example, for the rotation group em two-dimensions, a function $f(\theta) = e^{ia\theta}$ is single valued if a is integer, t-valued if a is rational and multi-valued if a is irrational. The difficult related to multiply valued function and irreps is overcome by considering *universal covering group.*

A *covering group* of a topological group H is a covering space G of H such that G is a topological group and the covering map $f : G \to H$ is a continuous group *homomorphism*. The group H always contains a discrete invariant subgroup N such that

$$G \simeq H/N.$$

From last example, the functions $e^{ia\theta}$ will be single valued on H. Every irrep of G, single or multi-valued, is single valued on H (and then all definitions of multiplication and the orthogonality theorems hold).

A *double covering group* is a topological double cover in which H has two in G, and includes, for instance, the orthogonal group. If G is a covering group of H then they are *locally isomorphic*[1].

A universal covering group is always simple connected, any closed curve in this group can be shrunk or contracted to a point. If the group is already simple connect, its universal covering group is itself. The universal cover is always unique and always exist.

It easy to see that the rotation group SO(3) has as a universal cover the group SU(2)[2] which is isomorphic to the group to Sp(2). The group SO(N) has a double cover which is the spin group Spin(N) and for $N \geq 3$, the spin group is the universal cover of SO(N). A closed curve contractible to a point has components in spinor representation that are single valued (a closed curve that is not contractible to a point can be defined as double-valued). For $N \geq 2$ the universal cover of the special linear group SL(N, R) is not a matrix group, i.e. it has no faithful finite-dimensional representations. Besides the fact that Sl(2,C) is simple connected, the group Gl(2,C) is not simple connect, and one can see it remembering that Gl(2,C) = U(1)Sl(2,C), where the unitary group, which is only a phase, is infinitely connected.

For instance, the annulus and the torus are infinity connected. The sphere ($N > 1$) is simple connected. The simple circumference S_1 is infinite connected. The N-euclidian space is simple connected even when removing some single points from space.

If one can set up a 1-1 continuous correspondence between the points of two spaces, then they have the same connectivity. For example, $S_N, N \geq 2, \sum_{i=1}^{N} x_i^2 = 1$, is simple connected since can use *stereographic projection* to set a 1-1 correspondence with the $(N-1)$-dimension euclidian space.

[1] The Lie algebra are isomorphic but the groups are locally isomorphic: the proprieties of the groups are globally different but locally isomorphic.

[2] Any 2×2 hermitian traceless matrix can be written as $X = \vec{x}\vec{\sigma}$. For any element U of SU(2), $X' = UXU^\dagger$ is hermitian and traceless so $X' = \vec{x}'\vec{\sigma}$ and tr $X^2 =$ tr X^2. Thus $\vec{x}$ rotate into $\vec{x}'$ and we can associate a rotation to any U. Since U and $-U$ are associated to the same rotation, this gives a double covering of SO(3) by SU(2).

Example: Addition Group for S_1 and R

Let us construct the sum of the infinitely connected group S_1 and the line R, which is simple connected,

- R: $-\infty < x < \infty$.
- $S_1 : e^{2\pi ix}$ with $0 \leq \theta < 1$.

The map is in the form $R \to S_1$, which means $x \to e^{2\pi i\theta}$, which is the group U(1). From this example, we resume covering group relation on table 10.1.

$H \to G$
$H/N \simeq G$
The kernel N lies in the center of H and its a invariant subgroup.

Table 10.1: Diagram for covering groups.

A trivial example is the case of SU(2), which center is diag(1,-1), this is exactly Z_2.

Example: $SU(2) \simeq S_3$

The elements of SU(2) are given by g,

$$g = e^{\frac{i}{2}\omega_i\sigma_j} = cos(\frac{\omega}{2}) + \frac{i\omega}{2}sin\frac{\omega}{2},$$

with proprieties $g^\dagger g = 1$ and det $g = 1$. All points at $\omega = 2\pi$ has to be identified. The surface of B_3 is a S_2. One wraps it and the point in the bottom is identified as the superficies $\omega = 2\pi$.

Isomorphism

It is a 1-1 mapping f of one group onto another, $G_1 \to G_2$, preserving the multiplication law. It has the same multiplication table. For example, every finite group is isomorphic to a permutation group. A relation of locally isomorphic groups is shown on table 10.2.

$SU(2)$	$\simeq$	$SO(3)$
$SU(1,1)$	$\simeq$	$SO(2,1)$
$SU(2) \times SU(2)$	$\simeq$	$SO(4)$
$Sl(2,C)$	$\simeq$	$SO(3,1)$
$SU(2) \times Sl(2,2)$	$\simeq$	$SO(4)^*$
$USP(4)$	$\simeq$	$SO(5)$
$USP(2,2)$	$\simeq$	$SO(4,1)$
$Sp(4,R)$	$\simeq$	$SO(3,2)$
$SU(4)$	$\simeq$	$SO(6)$
$SU(4)^*$	$\simeq$	$SO(5,1)$
$SU(2,3)^*$	$\simeq$	$SO(4,2)$ (AdS/CFT)
$Sl(4,R)$	$\simeq$	$SO(3,3)$
$Sl(3,1)$	$\simeq$	$SO(6)$

Table 10.2: Table of covering groups for the Lie groups.

Homomorphism

It is a 1-1 mapping f to the same point, $G \to G'$, preserving the metric structure. The image $f(G)$ forms a subgroup of G', the kernel K forms a subgroup of G.

The *first isomorphism theorem* says that any homomorphism $f(G)$ with kernel K is isomorphic to G/K. The kernel of a homorphism is an invariant subgroup.

10.2 Invariant Integration

Groups can be seen as curved manifolds and the *Haar measure* is a way to assign an invariant volume to subsets of locally compact topological groups and define an integral for functions on these groups. It possible to have a measure when

$$\sum_{g\in G} f(g) = \sum_{g} f(g \circ g),$$

and the left measure is then

$$\int f(\alpha)_{\mu_L}(\alpha) d_\alpha = \int f(\beta \circ \alpha)_{\mu_L}(\alpha) d\alpha. \tag{10.2.1}$$

We want to associate to a set of elements in the neighborhood of a element A a volume (measure) τ_A such that the measure of the elements obtained from these elements by left translation with B is the same, $d\tau_A = d\tau_{BA}$.

The generalization of this for continuous groups is giving by writing the elements as $g(\alpha) = e^{\alpha^\mu T_\mu}$. Clearly $g(\alpha)g(\beta) = g(\alpha \circ \beta)$, which is the group composition law and it is exactly (10.2.1). In this representation, the unitary element e is g(0). The invariant integration has also the propriety of internal translation invariance,

$$\int f(x)dx = \int f(x-q)dx.$$

The right invariant measure is given by

$$\int_{g\in\mu} f(\alpha)\mu_R(\alpha)d\alpha = \int f(-\beta\circ\alpha)\mu_R(\alpha)d\alpha.$$

For compact, for infinite and for discrete groups, $\mu_L = \mu_R$, the left measure is equivalent to the right. A pragmatic way of finding the measure of a group is defining a *density function* $\rho(a)$ such that

$$d\tau_A = \rho(A)dA = \rho(BA)dAB = d\tau_{BA}.$$

Isomorphic groups always have the same density function. In the neighborhood of the identity, we can make

$$dB = J(B)dA,$$

where $J(B)$ is the *jacobian*, i.e. the left measure can be defined as the determinant of the matrix of all changes (the jacobian).

Trivial Examples

For the group $g' = a + g$, one defines the function $\phi(a, b) = a + b$. The jacobian is given by

$$\frac{\partial\phi(a,b)}{\partial a}\Big|_{a=0} = 1.$$

Then $J(b) = 1, \rho(b) = 1$ and the measure is $\int dbf(b)$.

For the group $g' = ag$, one defines the function $\phi(a, b) = ab$ and

$$\frac{\partial\phi(a,b)}{\partial a}\Big|_{a=1} = b.$$

Then $J(b) = b, \rho(b) = \frac{1}{b}$ and the measure is $\int \frac{1}{b}dbf(b)$.

A Pathological Example[3]

For the group given by the representation element

$$g(\alpha) = \begin{pmatrix} e^{\alpha^1} & \alpha^2 \\ 0 & 1 \end{pmatrix}.$$

one has, from $g(\beta)g(\alpha) = g(\beta \circ \alpha)$,

$$g(\alpha)g(\beta) = \begin{pmatrix} e^{\beta^1} & \beta^2 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} e^{\alpha^1} & \alpha^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\alpha^1+\beta^1} & e^{\beta^1}\alpha^2 + \beta^2 \\ 0 & 1 \end{pmatrix}.$$

Writing as

$$\psi^1(\beta,\alpha) = \beta^1 + \alpha^1,$$
$$\psi^2(\beta,\alpha) = e^{\beta^1}\alpha^2 + \beta^2,$$

we have

$$\begin{pmatrix} e^{\psi^i} & \psi^2 \\ 0 & 1 \end{pmatrix}.$$

The left measure is then given by

$$\frac{\partial \psi^{1,2}(\beta,\alpha)}{\partial \alpha^2}\Big|_{\alpha_2=1,\alpha_1=0} = e^{\beta^1} \to \mu_L(\beta) = e^{\beta^1},$$

the right measure is given by

$$\frac{\partial \psi^{1,2}(\beta,\alpha)}{\partial \alpha^1}\Big|_{\alpha_2=0,\alpha_1=1} = 1 \to \mu_R(\beta) = 1.$$

Therefore we see that these two measures are different for this non semi-simple space.

Theorem: The left measure is equal to the right measure if the structure constant of the Lie algebra has trace equal to zero $f^{\mu}_{\mu\alpha} = 0$.

[3]Exercise proposed by Prof. van Nieuwenhuizen.

Invariant measure in Lie Groups

In SU(2) (also Sl(2,R), Sl(2,C) or Gl(2,R)), one has

$$g = a_0 + a_i^k \sigma_k = e^{i\frac{\omega^i \sigma_i}{2}},$$

where

$$\sigma_0^2 + \sum \sigma_k^2 = 1.$$

The Haar measure can then be written as

$$\int f(\alpha)\mu_L(\alpha) d_\alpha^1 d_\alpha^2 d_\alpha^3.$$

Appendix A

Table of Groups

Group	Name	Dim Def irrep	Section
S_n	Symmetric group	N, finite	1.4
Z_n, C_n	Cyclic group	N, finite	1.4
A_n	Alternating group	N, finite	1.4
D_{2N}	Dihedral group	2N, finite	1.4
U(N)	Unitary group	N, Infinite	3
SU(N)	Special Unitary group	N, Infinite	3
SO(2N)	Special Orthogonal group	2N, Infinite	4
Sp(2N)	Symplectic group	2N, Infinite	7
GL(N,C)	General Linear group	N, Infinite	10.1
SL(N,C)	Special Linear group	N, Infinite	10.1

Table A.1: Summary of groups.

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