PHY 688: The Application of Simulation in Astrophysics
Homework #3
Due: Mon. Mar. 27, 2006

1. Consider the thermal diffusion equation:

$$T_t = \kappa T_{xx} \tag{1}$$

Here, κ is the thermal diffusivity. This is not a hyperbolic equation, but we can still write finite-difference approximations to this equation.

a. Create an explicit discretization of Eq. 1 using a first order accurate time derivative and a second-order accurate second spatial derivative. Using von Neumann stability analysis, find the stability criteria for this method.

b. Now create an implicit discretization of Eq. 1 using the same order differences for the time and space derivatives. Again, using von Neumann stability analysis, find the stability criteria.

2. In this problem we look at several first-order finite-difference methods for Burger's equation:

$$u_t + uu_x = 0 \tag{2}$$

In conservative form, this appears as

$$u_t + \left[\frac{1}{2}u^2\right]_x = 0 \tag{3}$$

a. Using first-order upwinding, we can difference Eq. 2 as

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n) \tag{4}$$

This is valid as long as $u_i^n > 0$. Write a program to solve this equation with initial conditions:

$$u(x, t = 0) = \begin{cases} 2 & \text{if } x < 0.5 \\ 1 & \text{if } x \geq 0.5 \end{cases} \tag{5}$$

and outflow boundary conditions. What type of solution do you see? Also run it with:

$$u(x, t = 0) = \begin{cases} 1 & \text{if } x < 0.5 \\ 2 & \text{if } x \geq 0.5 \end{cases} \tag{6}$$

Now what does the solution look like?

This is a non-linear equation—make sure you use the proper CFL condition for this method.

b. Now difference Eq. 3 as

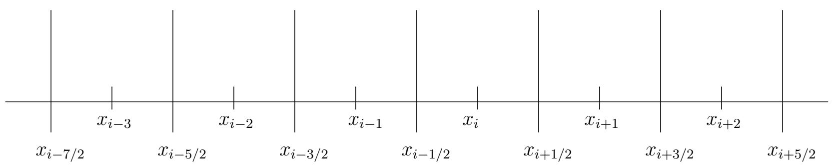
$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right) \tag{7}$$

Write a program to solve this, and run it on the same initial data as in part **a**.

How do the solutions differ for the two discretizations? One of the initial conditions generates a shock—measure its speed for each method. How does the shock speed differ with resolution? Which method gets the shock speed correct? Why? Present your results at 3 different grid resolutions. **Attach your codes for each method.**

Hint: You may want to start with the `ftcs.f90` code on the class website.

3. Consider discretizing a function $f(x)$ in one-dimension by breaking x into a collection of intervals (cells) as shown below.



Here, x_i marks the center of the cell i, and the left and right edges are denoted as $x_{i-1/2}$ and $x_{i+1/2}$ respectively. The cell width, δ is then just $x_{i+1/2} - x_{i-1/2}$.

A popular way to represent the function, $f(x)$, on this grid is to assign the average value of $f(x)$ to each cell, where the average is

$$\langle f \rangle_i = \frac{1}{\delta} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x) dx \tag{8}$$

This is called a *finite-volume* discretization.

a. Show that to second-order accuracy in the cell-spacing, δ, the average value of the function in interval $[x_{i-1/2}, x_{i+1/2}]$ is simply $f(x)$ evaluated at the midpoint of the cell, x_i. That is, prove,

$$\langle f \rangle_i = f(x_i) + O(\delta^2) \tag{9}$$

b. Consider a three zone averages, $\langle f \rangle_{-\delta}$, $\langle f \rangle_0$, $\langle f \rangle_\delta$ corresponding to the cells with centers $-\delta$, 0, and δ respectively. spacing.

Show that the coefficients of the polynomial $f(x) = ax^2 + bx + c$ with zone averages $\langle f \rangle_{-\delta}$, $\langle f \rangle_0$, $\langle f \rangle_\delta$ is

$$f(x) = \frac{\langle f \rangle_\delta - 2\langle f \rangle_0 + \langle f \rangle_{-\delta}}{2\delta^2} x^2 + \frac{\langle f \rangle_\delta - \langle f \rangle_{-\delta}}{2\delta} x + \frac{-\langle f \rangle_\delta + 26\langle f \rangle_0 - \langle f \rangle_{-\delta}}{24} \tag{10}$$

This process is called *reconstruction.*

Hint: We have three unknowns, a, b, and c, and three equations,

$$\langle f \rangle_\alpha = \frac{1}{\delta} \int_{\alpha-\delta/2}^{\alpha+\delta/2} (ax^2 + bx + c) dx \tag{11}$$

with $\alpha = -\delta, 0, \delta$.