PHY 688: The Application of Simulation in Astrophysics

Homework #2

Due: Fri. March 10, 2006

0. In a few weeks, we will begin running multidimensional hydrodynamic simulations using the FLASH Code. Please go to `http://flash.uchicago.edu/website/coderequest/` and request a copy of the code. After agreeing to the license agreement, you will get instructions on how to download the code.

1. (7 pts.) In our discussion on shocks, we derived an expression for the shock speed for a right-moving shock as

$$S = u_R + c_R \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_\star}{p_R}\right) + \left(\frac{\gamma-1}{2\gamma}\right)} \tag{1}$$

by moving into the frame of the shock

$$\begin{aligned} \hat{u}_\star &= u_\star - S && (2)\\ \hat{u}_R &= u_R - S && (3) \end{aligned}$$

and applying the jump conditions:

$$\begin{aligned} \rho_\star \hat{u}_\star &= \rho_R \hat{u}_R && (4)\\ \rho_\star \hat{u}_\star^2 + p_\star &= \rho_R \hat{u}_R^2 + p_R && (5)\\ \hat{u}_\star \left(\rho_\star e_\star + \frac{1}{2}\rho_\star \hat{u}_\star^2 + p_\star\right) &= \hat{u}_R \left(\rho_R e_R + \frac{1}{2}\rho_R \hat{u}_R^2 + p_R\right) && (6) \end{aligned}$$

Here, we will derive some related expressions that will be very useful later.

a. We begin by seeking an expression for the density ratio in terms of the pressure ratio. First, write an expression for the enthalpy ($h = e + p/\rho$) as

$$\frac{1}{2}\hat{u}_\star^2 + h_\star = \frac{1}{2}\hat{u}_R^2 + h_R \tag{7}$$

b. Now using Eqs. 4 and 5, eliminate the velocities in Eq. 7 in favor of the pressures and densities.

$$h_\star - h_R = \frac{1}{2}(p_\star - p_R)\frac{\rho_\star + \rho_R}{\rho_\star \rho_R} \tag{8}$$

and switch back to internal energy to get

$$e_\star - e_R = \frac{1}{2}(p_\star + p_R)\frac{\rho_\star - \rho_R}{\rho_\star \rho_R} \tag{9}$$

This relation is called the *Hugoniot equation* for the shock. It relates all the points in the (ρ, p) plane that can be connected by a shock wave.

c. Finally, using the equation of state, $p = \rho e(\gamma - 1)$, replace e in Eq. 9 and solve for the density ratio in terms of the pressure ratio to yield:

$$\frac{\rho_\star}{\rho_R} = \frac{\frac{p_\star}{p_R} + \frac{\gamma-1}{\gamma+1}}{\frac{p_\star}{p_R}\frac{\gamma-1}{\gamma+1} + 1} \tag{10}$$

This is LeVeque Eq. 14.53.

d. Now we want to compute the post-shock velocity, $u_\star$. From the Eq. 4 and Eqs. 2 and 3, show that

$$u_\star = \left(1 - \frac{\rho_R}{\rho_\star}\right) S + \frac{\rho_R}{\rho_\star} u_R \tag{11}$$

e. Substitute in Eq. 1 and Eq. 10 to get the expression for the post-shock speed in terms of the pressure ratio:

$$u_\star = u_R - \frac{2c_R}{\sqrt{2\gamma(\gamma-1)}} \frac{1 - \frac{p_\star}{p_R}}{\sqrt{1 + \frac{\gamma+1}{\gamma-1}\frac{p_\star}{p_R}}} \tag{12}$$

This is LeVeque Eq. 14.55. We will make use of this expression in class when computing the solution to the Euler equations.

2. (1 pt.) Write a program to determine the *machine epsilon*, ϵ. Here ϵ is the smallest number such that $1 + \epsilon$ is distinguishable from 1.

3. (7 pts.) In class we derived the Lane-Emden equation for stellar structure.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\phi}{d\xi}\right) = -\phi^n \tag{13}$$

In this problem, we are going to work out the solution, using numerical algorithms for ODEs. First, we split Eq. 13 into two first-order equations with the substitution $f = \phi$ and $g = \phi'$, yielding

$$\frac{df}{d\xi} = g \tag{14}$$

$$\frac{dg}{d\xi} = -\frac{2}{\xi} g - f^n \tag{15}$$

with boundary conditions $f(0) = 1$ and $g(0) = 0$.

a. Note that at $\xi = 0$, the righthand side of Eq. 15 blows up. By Taylor expanding ϕ, using Eq. 13, and noting that $\phi(\xi) = \phi(-\xi)$, show that

$$\phi = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \ldots \quad \text{for } |\xi| \ll 1 \tag{16}$$

b. With this expansion, we can rewrite Eq. 15 near $\xi = 0$ so that it behaves well. Using both the

Euler and two Runge-Kutta methods we described in class, integrate this system of equations, with n = 1, 1.5, and 3.

Using a constant step size, you will see that at some point, $f = \phi$ will go negative—this marks the outer edge of the star. Numerically, the integration scheme will not be well behaved for negative f, so we need to terminate the integration right at the radius of the star.

Since $d^2\phi/d\xi^2 > 0$ for $\xi \gg 0$, the density is concave and always monotonically decreasing. This means that the intersection of any tangent to $\phi(\xi)$ with the ξ axis will yield an estimate of the radius, $r^\star$ that is always less than the true radius r. Using this behavior, estimate the location of the radius of the star given the current integration point, ξ. (The result is the same as using Newton's method, but the concavity tells us that the estimated radius is always less than the true radius).

In your integration routines, estimate the radius of the star and keep the stepsize constant unless the step will take you beyond the star's radius. In that case, decrease the step size so that the next step takes you to $r^\star$. Continue this process, until the stepsize is below $\epsilon \sim 10^{-12}$.

Plot your solutions on a single set of axes and list the radius of the star.

c. Discuss the convergence properties of your integration routines for a variety of stepsizes. What is a good measure for the accuracy of the integration? Note, that the example in class used a constant stepsize throughout the entire integration, whereas your stepsize will decrease at the very end of the calculation.

Demonstrate the convergence by plotting your error versus step size on a log-log plot for both integration methods. Can you identify where truncation error dominates the error and where roundoff error comes into play?

$\star$ Include a copy of your program with your results.