PHY 688: The Application of Simulation in Astrophysics
Homework #1
Due: Wed Feb. 15, 2006

1. In class, we derived the Euler equations,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U}) &= 0 && (1) \\ \frac{\partial (\rho \vec{U})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U} \vec{U}) + \vec{\nabla} p &= \rho \vec{g} && (2) \\ \frac{\partial (\rho E)}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U} E + p \vec{U}) &= \rho \vec{U} \cdot \vec{g} && (3) \end{aligned}$$

where here, we've included the gravitational source terms. The specific energy, E can be written as the sum of internal and kinetic energies:

$$\rho E = \rho e + \frac{1}{2} \rho \vec{U} \cdot \vec{U} \qquad (4)$$

By substituting Eq. 4 into Eq. 3, show that the internal energy evolution is governed by:

$$\frac{\partial (\rho e)}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U} e) + p \vec{\nabla} \cdot \vec{U} = 0 \qquad (5)$$

Rewrite this in Lagrangian form as:

$$\rho \frac{De}{Dt} + p \vec{\nabla} \cdot \vec{U} = 0 \qquad (6)$$

and use the mass continuity equation to rewrite this as

$$\frac{De}{Dt} + p \frac{Dv}{Dt} = 0 \qquad (7)$$

where $v \equiv 1/\rho$ is the specific volume.

This is the first law of thermodynamics (with no changes in entropy).

2. We showed that the one-dimensional Euler equations can be written in primitive variable form as

$$W_t + A(W) W_x = 0 \qquad (8)$$

where

$$W = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \qquad A(W) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{pmatrix} \qquad (9)$$

a. Show that the eigenvalues of A are $\lambda_1 = u - c$, $\lambda_2 = u$, and $\lambda_3 = u + c$.

b. Compute the right eigenvectors, $\mathbf{r}$, defined as

$$A\mathbf{r} = \lambda \mathbf{r} \tag{10}$$

for each eigenvalue.

c. Compute the left eigenvectors, $\mathbf{l}$, defined as

$$\mathbf{l}A = \lambda \mathbf{l} \tag{11}$$

for each eigenvalue.

3. In this problem, we will write the momentum equation in a different form, which we will use in class to demonstrate some new properties of the Euler equations.

a. Prove the vector identity:

$$(\vec{U} \cdot \vec{\nabla})\vec{U} = (\vec{\nabla} \times \vec{U}) \times \vec{U} + \frac{1}{2}\vec{\nabla}|\vec{U}|^2 \tag{12}$$

where $\vec{U}$ is any arbitrary vector.

Hint: It may be best to work in index notation, and recall that

$$\vec{\nabla} \times \vec{U} = \epsilon_{ijk}\partial_j U_k$$

where ϵ_{ijk} is the permutation (or Levi-Civita) tensor with the following properties:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji}$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

b. Using this identity, rewrite the momentum equation as:

$$\frac{\partial \vec{U}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}|\vec{U}|^2\right) + (\vec{\nabla} \times \vec{U}) \times \vec{U} + \frac{1}{\rho}\vec{\nabla}p = \vec{g} \tag{13}$$

Project Each student will give a 15 minute (including questions) presentation on how computation is applied in their field of interest. You can pick any field related to astrophysics (preferable) or physics in general. Select 1 or more papers that present results of the application of hydrodynamic simulation on the field. These paper(s) should discuss the relevant physics, what equations were solved, an summary of the method used to solve them, and the results.

By Wed. Feb 15, 2006, you should have picked your topic and let me know what you will talk on. The talks will likely be in the last week of Feb. or the first week of March. Please discuss any potential topics you might have with me. A good source of papers is astro-ph (http://www.arxiv.org/list/astro-ph/new).