Final Exam

You can attempt any combination of complete problems worth up to 60 points. You are not allowed to collaborate with other students. The exam can be either handed in person by 4pm on Friday, Dec 16, or scanned/photographed and emailed to me by 1am on Dec 17. Hardcopies identical to any scanned/photographed exams should be handed to me by 10am on Dec 19.

1. (10 points) Equilibria
   a. (3 points) An intermediate mass star has evolved on the main sequence and become inhomogeneous: crudely it is all hydrogen exterior to some mass shell \( M_r = M_d \) and all helium interior to \( M_d \). Find the ratio of interior density divided by exterior density at the density discontinuity \( M_d \).
   b. (7 points) The star has radiative outer layers. Assume that \( T = 0 \) and \( P = 0 \) at its surface \( r = R \), and that \( M_r \approx M \) in the outer layers. Find an expression for the dependence of \( T \) on \( r \) close to \( r = R \), assuming electron scattering opacity (i.e., \( \kappa = 0.2 \( (1+X) \)).

2. (30 points) Nuclear Reactions
   When it begins carbon burning, a model 15 \( M_\odot \) star has surface photon luminosity \( L_\gamma = 8 \times 10^4 \) \( L_\odot \) and central density \( \rho_c = 5 \times 10^4 \) g cm\(^{-3} \). 10\(^4\) years later, carbon is depleted and oxygen burning begins. The surface luminosity is the same, but the central density has risen to \( \rho_c = 7 \times 10^6 \) g cm\(^{-3} \). Oxygen is depleted in only 4 years, and silicon burning begins.
   a. (3 points) Given that the photon luminosity of the star is the same in both the carbon and the oxygen burning phases, that similar masses of stellar material are fused in both phases, and that both carbon burning and oxygen burning liberate about 13 MeV per pair of nuclei fused, ‘prove’ that both burning phases should have almost the same lifetime. What may account for the factor of 2500 difference in the lifetimes of the two phases?
   b. (6 points) Derive the result that the rate of nuclear reactions between nuclei of charge (in electron units) \( Z_1 \) and \( Z_2 \) varies with temperature as a slowly varying prefactor, times \( e^{-\tau} \), where
      \[
      \tau = 4.25 \left( \frac{Z_1^2 Z_2^2 A}{T/10^9 \text{K}} \right)^{1/3},
      \]
   and \( A = A_1 A_2/(A_1 + A_2) \) is the reduced mass in atomic mass units. Your derivation need not produce the factor of 4.25 exactly; i.e., you may leave a dimensionless ‘factor of order unity’ unevaluated.
   c. (2 points) Evaluate \( \tau \) for the center of the 15 \( M_\odot \) star at the beginning of the C burning \( (^{12}\text{C} + ^{12}\text{C}) \) and at the beginning of the O burning \( (^{16}\text{O} + ^{16}\text{O}) \), assuming \( T = 10^9 \) K in both cases (you will get better estimates for \( T \) below).
   d. (3 points) Evaluation of the slowly varying prefactors gives total energy generation rates for carbon burning as
\[
\varepsilon_c = 5 \times 10^{43} \rho T_9^{2/3} X_c^2 e^{-\tau_c} \text{ erg g}^{-1} \text{ s}^{-1}
\]
and for oxygen burning as
\[
\varepsilon_o = 2 \times 10^{53} \rho T_9^{2/3} X_o^2 e^{-\tau_o} \text{ erg g}^{-1} \text{ s}^{-1},
\]
where \(T_9 = T/10^9 \text{ K}\), \(\rho\) is in g cm\(^{-3}\), and \(X_c\) and \(X_o\) are respectively the mass fractions of C and O. Using your best guesses for the mass involved in burning and the total luminosities being emitted by the macroscopic star, estimate the energy generation rate \(\varepsilon\) as \(L_{\text{tot}}/M(\text{burning})\).

e. (3 points) Given just these estimates for \(\varepsilon\), and the central densities quoted for the model at the beginning of carbon and oxygen burning, using the equations above derive the central temperatures of carbon-burning and oxygen-burning stars.

f. (3 points) Figure 1 (from Itoh et al. 1989, ApJ, 339, 354) shows contours of constant neutrino energy-loss rates \(Q_\nu = \varepsilon_\nu \rho\) due to pair, photo-, plasma and bremsstrahlung processes. The contours are labeled by \(\log_{10} Q\), where \(Q\) is in erg \(s^{-1} \text{ cm}^{-3}\). Check that the neutrino losses \(\varepsilon_\nu\) for the central density and temperature you found for the carbon-burning and oxygen-burning phases are consistent with the total \(\varepsilon\)'s you adopted to find the temperatures in part (e).

3. (10 points) Structure
   a. (3 points) State the basic equations of stellar structure without writing them down in detail. Give physical descriptions of how they are derived.
   b. (3 points) What are the boundary conditions? When are the radiative zero boundary conditions a good enough approximation?
   c. (2 points) How does convection change the temperature gradient?
d. (2 points) What is the effect of ionization on the adiabatic temperature gradient?

4. (10 points) Convection
   a. (3 points) Why is there a convection region near the surface of the sun?
   b. (2 points) Under what conditions is a convection region nearly isentropic?
   c. (2 points) Describe physically what happens to a fluid element during convection.
   d. (3 points) What is semi-convection, and what conditions does it occur?

5. (10 points) Evolution
   a. (3 points) Draw typical evolutionary tracks of a low-mass star and a high-mass star in the HR diagram. Identify and explain the various stages.
   b. (2 points) What is the Hayashi forbidden region, and why are no stars observed within it?
   c. (2 points) Why is neutral hydrogen (H I) opacity so sensitive to temperature?
   d. (2 points) What stars have He flashes, and what happens after the flash?
   e. (1 point) What are the energy sources on the horizontal branch, and which one is dominant?

6. (20 points) Spectral Lines
   For the following, assume that the stellar photosphere is simply a layer of cool non-emitting gas residing above a source of continuous radiation.
   a. (6 points) What factors determine how strong a spectral line will be? What factors determine how wide a spectral line will be? What factors determine how deep a spectral line will be?
   b. (7 points) Prove that the equivalent width of a weak spectral line is proportional to the abundance of the species causing the line.
   c. (7 points) Prove that the equivalent width of a strong spectral line is proportional to the square root of the abundance of the species causing the line. 
   (Hint: the emitted specific flux as a function of frequency $\nu$ is $F_{\nu}/F_{cont} = \exp(-\tau_\nu)$. Express $\tau_\nu$ via the element abundance and the atomic cross section $\sigma_a$ for the transition. Integrate $F_{\nu}/F_{cont}$ to obtain the equivalent width.)

7. (10 points) Opacities
   a. (3 points) What are the main radiative opacity sources in stellar atmospheres and interiors?
   b. (4 points) Sketch the dominant kinds of hydrogen opacity in stellar atmospheres as a function of atmospheric temperature. What is the dominant source of opacity for stars in the $7000 < T_{eff} < 10,000$ K range? Why?
   c. (3 points) Why does the metal abundance dramatically affect the electron pressure in the photosphere of a cool star, but not in a hot star?