Stellar Structure

Hydrostatic Equilibrium

Spherically symmetric Newtonian equation of hydrostatics:

$$dP/dr = -Gm\rho/r^2, \qquad dm/dr = 4\pi\rho r^2. \tag{1}$$

$m(r)$ is mass enclosed within radius r.

Conditions at stellar centers $Q = P + Gm^2/8\pi r^4$:

$$\frac{dQ}{dr} = \frac{dP}{dr} + \frac{Gm}{4\pi r^4}\frac{dm}{dr} - \frac{Gm^2}{2\pi r^5} = -\frac{Gm^2}{2\pi r^5} < 0 \tag{2}$$

$$Q\,(r \to 0) \to P_c, \qquad Q\,(r \to R) \to GM^2/8\pi R^4.$$

M and R are total mass and radius.

Central pressure P_c (Milne inequality):

$$P_c > \frac{GM^2}{8\pi R^4} = 4 \times 10^{14} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4 \text{dynes cm}^{-2}. \tag{3}$$

Average density is

$$\bar{\rho} = \frac{3M}{4\pi R^3} \simeq 1.4 \left(\frac{M}{M_\odot}\right) \left(\frac{R_\odot}{R}\right)^3 \text{g cm}^{-3}. \tag{4}$$

Estimate of T_c from perfect gas law:

$$T_c \simeq \frac{P_c \mu}{\bar{\rho} N_o} > 2.1 \times 10^6 \left(\frac{M}{M_\odot}\right) \left(\frac{R_\odot}{R}\right) \text{K}. \tag{5}$$

μ is mean molecular weight. T_c too low by factor of 7.

Better Estimate:

$$\rho = \rho_c \left[1 - (r/R)^2\right], \qquad M = (8\pi/15)\,\rho_c R^3.$$

$$P = P_c - \frac{4\pi}{3} G\rho_c^2 R^2 \left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{2}{5}\left(\frac{r}{R}\right)^4 + \frac{1}{10}\left(\frac{r}{R}\right)^6\right]. \tag{6}$$

$P(R) = 0$:

$$P_c = \frac{15GM^2}{16\pi R^4} = 3.5 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4 \text{dynes cm}^{-2}, \quad (7)$$

$$P = P_c \left[1 - \left(\frac{r}{R}\right)^2\right]^2 \left[1 - \frac{1}{2}\left(\frac{r}{R}\right)^2\right] = \frac{P_c}{2}\left(\frac{\rho}{\rho_c}\right)^2 \left(1 + \frac{\rho}{\rho_c}\right). \quad (8)$$

The central density is

$$\rho_c = \frac{15M}{8\pi R^3} = \frac{5}{2}\bar{\rho} \simeq 3.6 \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^3 \text{g cm}^{-3}, \quad (9)$$

and the central temperature becomes

$$T_c \simeq \frac{P_c \mu}{\rho_c N_o} \simeq 7.0 \times 10^6 \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) \text{K}. \quad (10)$$

Mean molecular weight:
Perfect ionized gas ($k_B = 1$)

$$P = T \sum_i (1 + Z_i)\, n_i \equiv \rho N_o T/\mu \equiv NT, \quad (11)$$

Z_i is charge of ith isotope. Abundance by mass of H, He and everything else denoted by X, Y, and $Z = \sum_{i>He} n_i A_i/(\rho N_o)$. Assuming $1 + Z_i \simeq A_i/2$ for $i >$He:

$$\mu = \left[2X + \frac{3}{4}Y + \sum_{i>He} \frac{n_i\,(1 + Z_i)}{\rho N_o}\right]^{-1} \simeq \frac{4}{2 + 6X + Y} = \frac{4}{3 + 5X - Z}. \quad (12)$$

Solar gas ($X = 0.75, Y = 0.22, Z = 0.03$) has $\mu \simeq 0.6$.

Number of electrons per baryon for completely ionized gas $Z_{i>He} \simeq A_i/2$:

$$Y_e = X + \frac{Y}{2} + \sum_{i>He} \frac{n_i Z_i}{\rho N_o} \simeq X + \frac{Y}{2} + \frac{Z}{2} = \frac{1}{2}(1 + X)\,. \quad (13)$$

The Virial Theorem

Position, momentum, mass of ith particle: $\vec{r}_i$, $\vec{p}_i$, m_i.

Newton's Law $\vec{F}_i = \dot{\vec{p}}_i$ with $\vec{p}_i = m_i \dot{\vec{r}}_i$:

$$\frac{d}{dt}\sum \vec{p}_i \cdot \vec{r}_i = \sum \left(\dot{\vec{p}}_i \cdot \vec{r}_i + \vec{p}_i \cdot \dot{\vec{r}}_i \right) = \frac{d}{dt}\sum m_i \dot{\vec{r}}_i \cdot \vec{r}_i = \frac{1}{2}\frac{d^2 I}{d^2 t}, \tag{14}$$

Moment of inertia: $I = \sum m_i \vec{r}_i^{\,2}$.
Static situation: $d^2 I/dt^2 = 0$.
Non-relativistic gas: $\sum m_i \dot{\vec{r}}_i^{\,2} = \sum \vec{p}_i \cdot \dot{\vec{r}}_i = 2K$.

Total kinetic energy:

$$K = \frac{1}{2}\sum \vec{p}_i \cdot \dot{\vec{r}}_i = -\frac{1}{2}\sum \dot{\vec{p}}_i \cdot \vec{r}_i = -\frac{1}{2}\sum \vec{F}_i \cdot \vec{r}_i = -\left(1/2\right)\Omega. \tag{15}$$

Sum is virial of Clausius. For perfect gas, only gravitational forces contribute, since forces involved in collisions cancel.

$$\sum \vec{F}_i^G \cdot \vec{r}_i = \sum_{pairs} \vec{F}_{ij}^G \cdot \left(\vec{r}_i - \vec{r}_j\right) = -\sum_{pairs} \frac{G m_i m_j}{r_{ij}} \equiv \Omega. \tag{16}$$

Ω is gravitational potential energy, $r_{ij} = |\vec{r}_i - \vec{r}_j|$.

Perfect gas with constant ratio of specific heats, $\gamma = c_p/c_v$:

$$K = (3/2)\, NT, \qquad U = (\gamma - 1)^{-1}\, NT, \qquad E = U + \Omega = U - 2K.$$

U is internal energy, E is total energy.

$$E = U + \Omega = U\left(4 - 3\gamma\right) = \Omega \frac{3\gamma - 4}{3\left(\gamma - 1\right)}. \tag{17}$$

For $\gamma = 4/3$, $E = 0$. $\gamma < 4/3$, $E > 0$, configuration unstable. $\gamma > 4/3$, $E < 0$, configuration stable and bound by energy $-E$.

Application: contraction of self-gravitating mass $\Delta\Omega < 0$. If $\gamma > 4/3$, $\Delta E < 0$, so energy is radiated. However, $\Delta U > 0$, so star grows hotter.

Relativistic gas: $\sum \vec{p}_i \cdot \dot{\vec{r}}_i = c \sum \vec{p}_i = K = -\Omega$.

Another derivation:

$$V dP = -\frac{1}{3}\frac{Gm}{r} dm = -\frac{1}{3} d\Omega, \tag{18}$$

where $V = 4\pi r^3/3$. Its integral is

$$\int V(r)\, dP = PV\Big|_0^R - \int P(r)\, dV = -\frac{1}{3}\Omega \tag{19}$$

from Eq. (18). Thus $\Omega = -3\int P(r)dV$.
Non-relativistic case:

$$P = 2\epsilon/3, \qquad \Omega = -2K, \qquad E = \Omega + K = \Omega/2.$$

Relativistic case similar to non-relativistic case with $\gamma = 4/3$:

$$P = \epsilon/3, \qquad \Omega = -K, \qquad E = 0.$$

The critical nature of $\gamma = 4/3$ is important in stellar evolution. Regions of a star which, through ionization or pair production, maintain $\gamma < 4/3$ will be unstable, and will lead to instabilities or oscillations. Entire stars can become unstable if the average adiabatic index drops close to 4/3, and this actually sets an upper limit to the masses of stars. As we will see, the proportion of pressure contributed by radiation is a steeply increasing function of mass, and radiation has an effective γ of 4/3. We now turn our attention to obtaining more accurate estimates of the conditions inside stars.

Polytropic Equations of State

The polytropic equation of state, common in nature, satisfies

$$P = K\rho^{(n+1)/n} \equiv K\rho^{\gamma'}, \tag{20}$$

n is the polytropic index and γ' is the polytropic exponent.

1) Non-degenerate gas (nuclei + electrons) and radiation pressure. If $\beta = P_{gas}/P_{total}$ is fixed throughout a star

$$P = \frac{N_o}{\mu\beta}\left[\frac{3N_o}{\mu\beta a}(1-\beta)\right]^{1/3}\rho^{4/3} \tag{21a}$$

$$T = \left[\frac{3N_o}{\mu\beta a}(1-\beta)\right]^{1/3}\rho^{1/3} \tag{21b}$$

Here μ and a are the mean molecular weight of the gas and the radiation constant, respectively. Thus $n = 3$.

2) A star in convective equilibrium. Entropy is constant. If radiation pressure is ignored, then $n = 3/2$:

$$s = \frac{5}{2} - \ln\left[\left(\frac{\hbar^2}{2mT}\right)^{3/2}\rho N_o/\mu\right] = \text{constant} \tag{22a}$$

$$P = \frac{\hbar^2}{2m}\left(\frac{\rho N_o}{\mu}\right)^{5/3}\exp\left(\frac{2}{3}s - \frac{5}{3}\right) = K\rho^{5/3}. \tag{22b}$$

3) An isothermal, non-degenerate perfect gas, with pairs, radiation, and electrostatic interactions negligible: $n = \infty$. Could apply to a dense molecular cloud core in initial collapse and star formation.

4) An incompressible fluid: $n = 0$. This case can be roughly applicable to neutron stars.

5) Non-relativistic degenerate fermions: $n = 3/2$. Low-density white dwarfs, cores of evolved stars.

6) Relativistic degenerate fermions: $n = 3$. High-density white dwarfs.

7) Cold matter at very low densities, below 1 g cm^{-3}, with Coulomb interactions resulting in a pressure-density law of the form $P \propto \rho^{10/3}$, i.e., $n = 3/7$.

Don't confuse polytropic with adiabatic indices. A polytropic change has $c = dQ/dT$ is constant, where $dQ = TdS$. An adiabatic change is a specific case: $c = 0$.

$$\gamma' = \frac{\partial \ln P}{\partial \ln V} = \gamma \frac{c_v \left(c - c_p\right)}{c_p \left(c - c_v\right)},$$

where the adiabatic exponent $\gamma = (\partial \ln P/\partial \ln V)|_s$. If $\gamma = c_p/c_v$, as for a perfect gas, $\gamma' = (c - c_p)/(c - c_v)$. In the adiabatic case, $c = 0$ and $\gamma' = \gamma$ regardless of γ's value.

Polytropes

Self-gravitating fluid with a polytropic equation of state is a polytrope, with

$$\Omega = -\int \frac{Gm\left(r\right) dm\left(r\right)}{r} = -\frac{3}{5-n}\frac{GM^2}{R} = -3\int PdV. \quad (23)$$

For a perfect gas with constant specific heats,

$$E = -\frac{3\gamma - 4}{\gamma - 1}\frac{1}{5-n}\frac{GM^2}{R}. \quad (24)$$

For the adiabatic case $n = 1/(\gamma - 1)$,

$$E = \frac{n-3}{5-n}\frac{GM^2}{R}. \quad (25)$$

For a mixture of a perfect gas and radiation,

$$U = \int \left[\frac{\beta}{\gamma - 1} + 3\left(1 - \beta\right)\right] PdV = \int \beta \frac{4 - 3\gamma}{\gamma - 1} PdV - \Omega. \quad (26)$$

For β = constant, Eq. (26) gives β times the result found in Eq. (24). A bound star has $E < 0$ and $\gamma > 4/3$. If $\gamma = 5/3$, $E = -(3\beta/7)(GM^2/R)$.

A nested polytrope has

$$\begin{array}{lll} P = K\rho^{1+1/n}; & \epsilon = nP & \rho < \rho_t \\ P = K\rho_t^{1/n-1/n_1}\rho^{1+1/n_1}; & \epsilon = n_1 P + (n - n_1)\, P_t & \rho > \rho_t. \end{array}$$

ρ_t and P_t are the transition density and pressure between indices n and n_1. ϵ is the energy density

$$\begin{aligned} E = {} & \frac{n-3}{5-n}\left[\frac{Gm^2}{R} - \frac{GM_t^2}{R_t}\right] + \frac{n_1-3}{5-n_1}\frac{GM_t^2}{R_t^2} \\ & + 3P_t\left[\frac{M_t}{\rho_t} - \frac{4\pi}{3}R_t^3\right]\left[\frac{n-1}{5-n} - \frac{n_1-1}{5-n_1}\right]. \end{aligned} \tag{27}$$

M_t and R_t are mass and radius interior to transition point. When ($n_1 \simeq 0$) and $n \simeq 3$,

$$E = -\frac{3}{5}\frac{GM_t^2}{R_t}. \tag{28}$$

This could apply to a proto-neutron star with relativistic electron gas up to ρ_t, and relatively stiff matter beyond. The energy depends on inner core size only.

Structure of polytropes and Lane-Emden equation:

$$r = A\xi, \quad \theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}, \quad A = \left[(n+1)\, K\rho_c^{1/n-1}/\,(4\pi G)\right]^{1/2}. \tag{29}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{30}$$

Boundary conditions for are θ=1 and $\theta' = d\theta/d\xi = 0$ at ξ=0. The radius is found from ξ_1 where $\theta(\xi) = 0$.

n	γ'	θ	ξ_1	$-\xi_1^2\theta_1'$	$-\xi_1/3\theta_1'$	$[4\pi(n+1)\theta_1'^2]^{-1}$
0	∞	$1-\xi^2/6$	$\sqrt{6}$	$2\sqrt{6}$	1	$3/8\pi$
1	2	$\sin(\xi)/\xi$	π	π	$\pi^2/3$	$\pi/8$
3/2	5/3		3.654	2.714	5.992	0.7704
2	3/2		4.353	2.411	11.40	1.638
3	4/3		6.897	2.018	54.19	11.05
3.25	17/13		8.019	1.950	88.15	20.36
4	5/4		14.97	1.797	622.3	247.5
5	6/5	$1/\sqrt{1+\xi^2/3}$	∞	$\sqrt{3}$	∞	∞

Analytic solutions exist in the following cases:

$$\theta = 1-\xi^2/6; \quad \xi_1=\sqrt{6} \quad n=0, \gamma'=\infty; \tag{31a}$$

$$\theta = \sin\xi/\xi; \quad \xi_1=\pi \quad n=1, \gamma'=2; \tag{31b}$$

$$\theta = 1/\sqrt{1+\xi^2/3}; \quad \xi_1=\infty \quad n=5, \gamma'=\frac{6}{5}. \tag{31c}$$

$$\text{Radius}: R = A\xi_1 \tag{32a}$$

$$\text{Mass}: M = -4\pi A^3\rho_c\xi_1^2\theta_1' \tag{32b}$$

$$\text{Density ratio}: \bar{\rho}/\rho_c = -3\theta_1'/\xi_1 \tag{32c}$$

$$\text{Central pressure}: P_c = GM/\left[4\pi\,(n+1)\,\theta_1'^2 R^4\right] \tag{32d}$$

$$K = \frac{G}{n+1}\left[4\pi\left(\frac{M}{-\xi_1^2\theta_1'}\right)^{n-1}\left(\frac{R}{\xi_1}\right)^{3-n}\right]^{1/n}. \tag{33}$$

For $n\to\infty$, we have the *isothermal* Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = e^{-\phi} = \frac{\rho}{\rho_c}. \tag{34}$$

$$\rho = K/2\pi G r^2; R=\infty; m = 2Kr/G \quad n=\infty, \gamma'=1. \tag{35}$$

For $n = 3$ mass does not depend on central density, but only on equation of state. For a relativistic degenerate electron gas,

$$P = \frac{\hbar c}{4}\left(3\pi^2\right)^{1/3}(nY_e)^{4/3}\,, \tag{36}$$

which implies a mass

$$M_{ch} = -4\pi\left(\frac{K}{G}\right)^{3/2} \times 2.018 = 5.76 Y_e^2 M_\odot. \tag{37}$$

This is the famous Chandrasekhar mass, the limiting mass of a white dwarf as $\rho_c \to \infty$. A degenerate mass larger than M_{ch} cannot be stabilized by electron pressure alone. For $T \neq 0$, the pressure has a small thermal component

$$P_{th} = \frac{T^2}{8\hbar c}\left(3\pi^2\rho Y_e\right)^{2/3}. \tag{38}$$

For a massive stellar core just prior to collapse, $T \simeq 0.7$ MeV and $\rho \simeq 6 \times 10^9$ g cm^{-3}, and the $P_{th}/P \simeq 0.12$, and the effective M_{ch} is $(1.12)^{3/2} = 1.19$ times larger. The negative Coulomb lattice pressure, which is about 4% of the total, lowers this. At densities in excess of 10^6 g cm^{-3}, electron capture decreases Y_e. For a ^{56}Fe white dwarf, the zero-temperature Chandrasekhar mass is only 1.17 M$_\odot$.

As the cores of massive stars evolve, there is a general tendency for "core convergence" to occur, *i.e.*, the evolved cores of all massive stars, regardless of mass, tend to be nearly M_{ch}. We see that this is a result of the general requirement for stability. In fact, there is a slight trend for more massive stars to have larger cores, but this can be traced to the higher entropies in these stars (recall that Eq. (5) predicts that $T \sim M/R$) and their larger effective Chandrasekhar masses.

Standard Model Stars – The Main Sequence

Those stars converting H to He. Standard model assumes $\beta =$ constant.

$$K = \left(\frac{N_o}{\mu\beta}\right)^{4/3}\left(\frac{3}{a}(1-\beta)\right)^{1/3}. \tag{39}$$

$$\begin{aligned} R &= 11.18\left[\frac{1-\beta}{\mu^4\beta^4\rho_c^2}\right]^{1/6} R_\odot \\ M &= 18\frac{\sqrt{1-\beta}}{\mu^2\beta^2} M_\odot \\ \rho_c/\bar{\rho} &= 54.2 \\ T_c &= 1.96\frac{\beta\mu M R_\odot}{M_\odot R}\times 10^7\mathrm{K}. \end{aligned} \tag{40}$$

For $n = 3$ polytrope, mass is independent of ρ_c and for given composition μ, is parametrized by β.

For Sun, with $M = 1\mathrm{M}_\odot$ and $\mu \simeq 0.6$:

$$\beta \simeq 1 - \left(\frac{\mu^2}{18}\right)^2 = 0.9996; \quad \rho_c = \frac{11.18^3}{18} = 76.7 \ \mathrm{g\ cm^{-3}}, \tag{41}$$

$$T_c = 1.307 \times 10^7 \ \mathrm{K}.$$

But $\mu_c > \bar{\mu}$ since some H→He has occurred.

Luminosity will depend upon nuclear energy generation $\dot{\epsilon}$ and transport (opacity κ).

For $T > 8 \cdot 10^6 \rho^{1/3.5}$ electron scattering dominates:

$$\kappa \simeq 0.4 Y_e \ \mathrm{cm^2\ g^{-1}}. \tag{42}$$

Where $T > 10^4$ K, κ dominated by bound-bound and bound-free processes:

$$\kappa \simeq 2.5 \cdot 10^{25} Z Y_e \rho T^{-3.5} \ \mathrm{cm^2\ g^{-1}}. \tag{43}$$

For $Z < 10^{-4}$ have free-free opacity:

$$\kappa \simeq 8 \cdot 10^{22} \left(1 - Z\right) Y_e \rho T^{-3.5} \ \ \mathrm{cm}^2 \ \ \mathrm{g}^{-1}.$$

The dependence $\kappa \propto \rho T^{-3.5}$ is known as Kramer's opacity. For $T < 10^4 K$, matter barely ionized:

$$\kappa \simeq 10^{-32} \left(Z/.02\right) \rho T^{10} \ \ \mathrm{cm}^2 \ \ \mathrm{g}^{-1}. \qquad (44)$$

Energy Transport:

$$L\left(r\right) = -4\pi r^2 \frac{4ac}{3\kappa_R \rho} T^3 \frac{dT}{dr}. \qquad (45)$$

$L(r)$ is luminosity, κ_R is the "Rosseland mean" opacity, averaged over frequencies. $(\kappa\rho)^{-1}$ is photon mean free path, $d(acT^4/4)/dr$ is radiation energy density gradient. Multiplied together gives energy flux, and multiplied by area $4\pi r^2$ gives net energy flow. Use hydrostatic equilibrium:

$$\frac{dP_r}{dP} = \frac{\kappa L\left(r\right)}{4\pi c G m\left(r\right)}. \qquad (46)$$

Luminosity function $\eta(r) = L(r)M/m(r)L$, with M and L totals. η is sharply peaked at origin.

$$d\left[\left(1 - \beta\right) P\right] = \frac{L}{4\pi c G M} \kappa\left(r\right) \eta\left(r\right) dP. \qquad (47)$$

$$L = \frac{4\pi c G M}{\bar{\kappa \eta}} \left(1 - \beta_c\right) \qquad (48)$$

$\bar{\kappa\eta}$ is a pressure average. (ssm: $\kappa\eta =$ cons.) With Eq. (43)

$$L_{ssm} = \left(4\pi\right)^3 \frac{4ac}{3\kappa_o \eta_c} \left(\frac{\mu_c \beta G}{4 N_o}\right)^{7.5} \frac{M^{5.5}}{\left(-\xi_1^2 \theta_1'\right)^{4.5}} \left(\frac{\xi_1}{R}\right)^{0.5}$$

$$\simeq .667 \frac{\left(\mu_c \beta_c\right)^{7.5}}{\eta_c Z Y_{e,c}} \left(\frac{M}{M_\odot}\right)^{5.5} \left(\frac{R_\odot}{R}\right)^{0.5} L_\odot. \qquad (49)$$

With Eq. (42), appropriate instead for more massive stars,

$$L \simeq 97.5 \frac{1}{\eta_c Y_{e,c}} (\beta \mu_c)^4 \left(\frac{M}{M_\odot} \right)^3 L_\odot. \tag{50}$$

For Sun: $\mu_c = 0.73, Y_{e,c} = 0.75$ (i.e., $X \simeq 0.5$ and $Y \simeq 0.5$),

$$L \simeq (2.8/\eta_c)\, L_\odot.$$

Alternatively, use proton-proton rate ($T_6 = T/10^6$):

$$\eta = 2.0 \times 10^6 \rho X^2 T_6^{-2/3} \exp\left(-33.8 T_6^{-1/3} \right) \frac{M_\odot}{L_\odot}. \tag{51}$$

With $\rho_c = 76.7$ g cm^{-3}, $T_{c,6} = 13.07$ from ssm η_c=4.05. Try using knowledge of polytropic structure. Assume ideal gas and

$$\dot\epsilon = \dot\epsilon_c \left(\frac{\rho}{\rho_c} \right)^\lambda \left(\frac{T}{T_c} \right)^\nu. \tag{52}$$

For polytrope, $\rho = \rho_c \theta^n$ and $T = T_c \theta$:

$$L = 4\pi A^3 \rho_c \dot\epsilon_c \int \xi^2 \theta^{2n\lambda+\nu} d\xi \simeq 4\pi A^3 \rho_c \dot\epsilon_c \sqrt{\frac{27\pi}{2}} (2n\lambda + \nu)^{-3/2}, \tag{53}$$

since $2n\lambda + \nu >> 1$. With $\theta \simeq \exp(-\xi^2/6), n = 3$,

$$\eta_c = \dot\epsilon_c \frac{M}{L} = -\left(\xi^2 \theta' \right)_1 \sqrt{\frac{2}{27\pi}} (2n\lambda + \nu)^{3/2} = 0.31\, (6\lambda + \nu)^{3/2}, \tag{54}$$

P-p cycle has $\lambda = 1, \nu \simeq 4$, so $\eta_c \simeq 9.8$.
CNO cycle has $\lambda = 1, \nu \simeq 20$, so $\eta_c \simeq 41$.

Scaling Relations for Standard Solar Model

For Kramer's opacity, $\kappa \propto Z(1+X)\rho T^{-3.5}$. For electron scattering, $\kappa \propto (1+X)$. Suggests

$$\kappa \propto (1+X)\, Z^u \rho^n T^{-s}, \tag{55}$$

with $u = 0, 1$. Similarly

$$\dot{\epsilon} \propto X^{2-m} Z^m \rho^\lambda T^\nu, \tag{56}$$

with $m = 0(1)$ for p-p (CNO) cycle. Using

$$L \propto RT^4/\kappa\rho \propto M\dot{\epsilon}, \qquad T \propto \mu\beta M/R, \qquad \rho \propto M/R^3, \tag{57}$$

we find

$$\begin{aligned} L &\propto M^{\alpha_M} (\mu\beta)^{\alpha_\mu} X^{\alpha_X} (1+X)^{\alpha_1} Z^{\alpha_Z}, \\ R &\propto M^{\beta_M} (\mu\beta)^{\beta_\mu} X^{\beta_X} (1+X)^{\beta_1} Z^{\beta_Z}, \\ T_{eff} \propto \left(L/R^2\right)^{1/4} &\propto M^{\gamma_M} (\mu\beta)^{\gamma_\mu} X^{\gamma_X} (1+X)^{\gamma_1} Z^{\gamma_Z}, \\ L &\propto T_{eff}^{\delta_T} (\mu\beta)^{\delta_\mu} X^{\delta_X} (1+X)^{\delta_1} Z^{\delta_Z}. \end{aligned} \tag{58}$$

With $i = (M, \mu, X, (1+X), Z, T)$, $\gamma_i = \alpha_i/4 - \beta_i/2$
$\delta_i = 2(\alpha_M\beta_i - \alpha_i\beta_M)/(\alpha_M - 2\beta_M)$, $D = \nu - s + 3(n+\lambda)$:

$i \rightarrow$	M	μ	T
$\alpha_i D$	$\nu(3+2n) + 9\lambda + 3n + s(2\lambda - 1)$	$7\nu + 3\lambda(4+s)$	0
$\beta_i D$	$\lambda + \nu + n - s - 2$	$\nu - 4 - s$	$2D$
$i \rightarrow$	X	1	Z
$\alpha_i D$	$m(3n-s) - u(3\lambda + s)$	$-(s+3\lambda)$	$(3n-s)(2-m)$
$\beta_i D$	$u + m$	1	$2-m$

	α_M	α_μ	α_X	α_1	α_Z	α_T
low	71/13	101/13	-2/13	-14/13	-16/13	0
high	3	4	0	-1	0	0
	β_M	β_μ	β_X	β_1	β_Z	β_T
low	1/13	-7/13	4/13	2/13	2/13	2
high	19/23	16/23	1/23	1/23	1/23	2
	γ_M	γ_μ	γ_X	γ_1	γ_Z	γ_T
low	69/52	87/52	-3/26	-9/26	-7/13	-2
high	31/92	15/23	-1/46	-25/92	-1/46	-2
	δ_M	δ_μ	δ_X	δ_1	δ_Z	δ_T
low	0	-4/3	44/69	8/23	68/69	284/69
high	0	-56/31	6/31	44/31	6/31	276/31

Values refer to low-mass ($\nu = 4, \lambda = 1, m = 0, u = 1, n = 1, s = 3.5$) or high-mass ($\nu = 20, \lambda = 1, m = 1, u = 0, n = 0, s = 0$) M-S stars.

1) As H consumed, X decreases and μ increases

$$\mu \simeq 4/\left(5X + 3\right)$$

and β is nearly constant. So L increases and T_{eff} increases; stars evolve *up* the main sequence. This explains why in globular clusters the M-S turnoff luminosity $>> L_\odot$ even though $M \leq \mathrm{M}_\odot$. Also, the early Sun was less luminous, and cooler, than present. If initial (present) $X = 0.75(0.7)$,

$$\frac{L_{today}}{L_{initial}} \simeq 1.4, \quad \frac{T_{eff,today}}{T_{eff,initial}} \simeq 1.11,$$

$$\frac{T_{c,today}}{T_{c,initial}} \simeq 1.09, \quad \frac{R_{today}}{R_{initial}} \simeq 0.96.$$

2) Stars on the p-p cycle ($\nu = 4$) have R nearly independent of M: $R \propto M^{1/13}$ for Kramer's opacity. For stars on the CNO cycle, however, $R \propto M^{11/15}$ for Kramer's opacity and $R \propto M^{19/23}$ for electron scattering opacity.

3) Population II stars are characterized by low metal compositions, $Z < 0.001$. For a given $T_{eff}, L \propto Z^{\delta_Z}$ dominates the composition dependence. The Population II M-S is shifted to lower L than the Population I M-S. Also, for a given M, $T_{eff} \propto Z^{\gamma_Z}$ implies a shift of the M-S to higher T_{eff}.

4) For a given M, $L \propto Z^{\alpha_Z}$, which is larger for Population II than for Population I stars. Stellar lifetimes $\tau \propto M/L$ are nearly $\propto Z$ since $\kappa \propto Z(1+X)$. Thus, lifetimes of Population II stars are substantially less than Population I for a given mass. This is observed in H-R diagrams of globular clusters.

Radiative Zero Solution

Besides the standard model, we could consider an idealized star in which the energy generation is uniform throughout, *i.e.*, $\eta(r) = 1$, and the equation of state is that of an ideal gas alone. If such a star is in radiative equilibrium, we can write

$$\frac{d\ln T}{d\ln P} = \frac{3L\kappa P}{16\pi acGMT^4} = \frac{3\kappa_o L}{16\pi acGM}\left(\frac{\mu}{N_o}\right)^m \frac{P^{m+1}}{T^{4+t+m}} \quad (1.1)$$

where the opacity is assumed to scale as

$$\kappa = \kappa_o \rho^n T^{-s}. \quad (1.2)$$

The *radiative zero solution* is obtained if $d\ln T/d\ln P$ is constant. Eq. (1.1) then implies that $d\ln T/d\ln P = (n+1)/(n+s+4)$ and

$$P \propto T^{(4+s+n)/(n+1)}; \quad P \propto \rho^{(4+s+n)/(s+3)}. \quad (1.3)$$

For a Kramer's opacity law ($n = 1, s = 3.5$) we find the effective polytropic index to be 3.25, and, from Eq. (1.1),

$$\begin{aligned} L &= (4\pi)^3 \frac{4ac}{3\kappa_o}\left(\frac{4\mu G}{17N_o}\right)^{7.5} \frac{M^{5.5}}{(-\xi_1^2\theta_1')^{4.5}}\left(\frac{\xi_1}{R}\right)^{0.5} \\ &\simeq 0.8\eta_c \left(\frac{\mu}{\mu_c}\right)^{7.5}\left(\frac{Y_{e,c}}{Y_e}\right) L_{ssm}. \end{aligned} \quad (1.4)$$

(Note that ξ_1 and θ_1' are evaluated for the $n = 3.25$ polytrope, and not the $n = 3$ polytrope as for the standard model). Had we used the Thomsen opacity ($n = s = 0$) instead, we would have just found Eq. (50) with β_c=1.

Completely Convective Stars

To conclude this section, we now consider the idealized completely convective star. This case is especially relevant to the pre-main sequence phase of stellar evolution. For a perfect gas, an $n = 3/2$ polytrope must result for constant entropy. We immediately find

$$\begin{aligned} \rho_c/\bar{\rho} &= 6, \\ T_c &= 1.2\frac{\mu M R_\odot}{M_\odot R} \times 10^7 \mathrm{K}, \\ P_c &= 8.7\frac{M^2 R_\odot^4}{M_\odot^2 R^4} \mathrm{erg\ \ cm}^{-3}. \end{aligned} \tag{1.5}$$

It is also clear that, dimensionally (*cf*. Eq. (32))

$$\begin{aligned} M\ &\propto K^{3/2}\rho_c^{1/2} \\ R\ &\propto K^{1/2}\rho_c^{-1/6} \propto KM^{-1/3} \propto e^{2s/3}\,(\mathrm{for\ \ fixed\ \ } M) \\ E\ &= \frac{3}{14}\frac{GM^2}{R} \propto M^{7/3}K^{-1} \quad \propto e^{-2s/3}\,(\mathrm{for\ \ fixed\ \ } M) \end{aligned} \tag{1.6}$$

where K is given by Eq. (22). In the last two equations, s is the entropy per baryon, not the temperature dependence of the energy generation rate.

It is straightforward to show that both the heat flux and luminosity vary as the 3/2 power of the difference of the actual temperature gradient from the purely adiabatic one (*e.g.*, Ref. **@Ref.Clayton@**, p. 257). Typically, near the outside of a star, this difference is only 10^{-6} of the temperature gradient itself. Therefore it is impossible to determine the luminosity from the transport equation as we did in the radiative case. But because radiation eventually escapes, the transport must become radiative just below the surface. Using the photospheric condition for the optical depth $\tau = \int \kappa\rho dr \simeq 2/3$ one may determine the surface temperature and hence the luminosity. Hydrostatic equilibrium can be rewritten as

$$dP/d\tau = -g/\kappa \tag{1.7}$$

where $g = GM/r^2 \simeq GM/R^2$, the surface gravity, is nearly constant throughout the thin surface region. As a zeroth approximation, we may write

$$P_p \simeq \frac{2}{3}\frac{g_p}{\kappa_p} = \frac{2}{3}GMR_p^{-2}\kappa_o^{-1}\rho_p^{-n}T_p^s \tag{1.8}$$

where the subscript p indicates photospheric values. Only if κ varies rapidly in the surface region will this result be inaccurate. Combining Eqs. (1.8) and (33), using values for a $n = 3/2$ polytrope, and employing the perfect gas law, we can find the photospheric temperature:

$$T_p = \left[\left(\frac{2GM}{3\kappa_o R_p^2}\right)^2 \left(\frac{\mu}{N_o}\right)^{3n+5} K^{3+3n}\right]^{1/Q} \propto \left(M^{3+n}R_p^{3n-1}\right)^{1/Q}, \tag{1.9}$$

where $Q = 5 + 3n - 2s$. The luminosity follows immediately from $L = 4\pi R_p^2 \sigma T_p^4$:

$$\begin{aligned} L =&\sigma\left[(4\pi)^{-n-5}\left(\frac{3\kappa_o}{5}\right)^4 \left(\frac{5N_o}{2G\mu}\right)^{10+6n} \left(-\xi_1^5\theta_1'\right)^{2+2n}\frac{T_p^{6+18n-4s}}{M^{6+2n}}\right]^{1/(3n-1)} \\ =&\sigma\left[(4\pi)^{13+11n-2s}\left(\frac{5}{3\kappa_o}\right)^8 \left(\frac{2\mu G}{5N_o}\right)^{20+12n}\frac{M^{12+4n}R_p^{6+18n-4s}}{\left(-\xi^5\theta_1'\right)^{4+4n}}\right]^{1/Q}. \end{aligned} \tag{1.10}$$

This relation shows the tremendous sensitivity of the luminosity to the photospheric temperature: typically in the low density surface regions, $s \approx -10$.

In general, a star is convective if its luminosity is large enough to force a superadiabatic temperature gradient. Thus, there must exist a minimum luminosity below which a star cannot be completely convective. A star in convective equilibrium has $d\ln T/d\ln P = 2/5 (n = 3/2)$, so from Eq. (33),

$$P_c = \left(\frac{N_o T_c}{\mu}\right)^{5/2} K^{-3/2}; \quad T_c = \frac{-2}{5\xi_1\theta_1'}\frac{\mu GM}{N_o R}. \tag{1.11}$$

On the other hand, a star in radiative equilibrium, from Eq. (1.4), satisfies, at the center,

$$\frac{d\ln T}{d\ln P} = \frac{3}{16\pi acG}\frac{\kappa_c P_c L\eta_c}{T_c^4 M}. \tag{1.12}$$

If the logarithmic temperature gradient at the center falls below 2/5, the star will cease to be completely convective. Therefore from the previous two equations, we find

$$\begin{aligned} L_{min} &= \frac{32\pi acGM}{15\eta_c}\frac{T_c^4}{\kappa_c P_c} \\ &= \frac{4ac}{3\eta_c\kappa_o}(4\pi)^{n+2}\left(\frac{2G\mu}{5N_o}\right)^{4+s}\frac{\xi_1^{s-3n}}{\left(-\xi_1^2\theta_1'\right)^{2+s-n}}\frac{M^{s-n+3}}{R^{s-3n}} \\ &= 271\frac{\mu^{7.5}\left(M/M_\odot\right)^{5.5}}{\eta_c\left(R/R_\odot\right)^{.5}}L_\odot \end{aligned} \tag{1.13}$$

where the last equality holds for Kramer's opacity. This can be compared with the luminosity from the standard solar model (for 1 $M_\odot$ and 1 $R_\odot$), which behaves on the physical variables in a similar way:

$$L_{min}/L_{ssm} = \left(\frac{8}{5\beta}\right)^{7.5} \left(\frac{\mu}{\mu_{ssm}}\right)^{7.5} \left(\frac{Y_{e,ssm}}{Y_e}\right) \times$$
$$\times \left(\frac{\xi_{1,3}^2 \theta'_{1,3}}{\xi_{1,3/2}^2 \theta'_{1,3/2}}\right)^{4.5} \left(\frac{\xi_{1,3/2} R_{ssm}}{\xi_{1,3} R}\right)^{0.5} \frac{\eta_{ssm}}{\eta_c}.$$

The numerical coefficient is equal to 6.518. We expect that $\eta_{ssm}/\eta_c \approx 2$, and $\mu/\mu_{ssm} \simeq 0.82$, so with $R \simeq 3R_\odot$ we find that $L_{min} \simeq 1.6 L_{ssm}$ for a solar-type star. This is larger than the actual minimum luminosity reached along the Hayashi track, but the star overshoots this minimum luminosity as it gradually becomes more and more radiative.

We will further explore pre-main sequence stars in the next chapter.