Classical Statistical Mechanics

A macrostate has N particles arranged among m volumes, with $N_i (i = 1 \ldots m)$ particles in the ith volume. The total number of allowed microstates with distinguishable particles is

$$W = \frac{N!}{\prod_i^m N_i!}; \quad \ln W = \ln N! - \sum_i^m \ln N_i! \, .$$

For a large number of particles, use Stirling's formula

$$\ln N! = N \ln N - N \, .$$

$$\ln W = N \ln N - N - \sum_i^m (N_i \ln N_i - N_i) \; .$$

The optimum state is the macrostate with the largest possible number of microstates, which is found by maximizing W, subject to the constraint that the total number of particles N is fixed ($\delta N = 0$). In addition, we require that the total energy be conserved. If w_i is the energy of the ith state, this is

$$\delta \left(\sum_i^m w_i N_i \right) = \sum_i^m w_i \delta N_i = 0 \, .$$

With these constraints, the minimization is

$$\delta \left[\ln \left(W - \alpha \sum_i^m N_i - \beta \sum_i^m w_i N_i \right) \right] = 0 \, .$$

$$\sum_i^m \left[\ln N_i - \alpha - \beta w_i \right] \delta N_i = 0 \, .$$

$$N_i = \alpha e^{\beta w_i} = \alpha e^{-w_i/kT} \, ,$$

which is the familiar Maxwell-Boltzmann, or classical, distribution function.

Quantum Statistical Mechanics

In the quantum mechanical view, only within a certain phase space volume are particles indistinguishable. The minimum phase space is of order h^3. Now denote the number of microstates per cell of phase space of volume h^3 as W_i. Then the number of microstates per macrostate is

$$W = \prod_i W_i \,.$$

Note we have to consider both the particles and the compartments into which they are placed. If the ith cell has n compartments, there are n sequences of $N_i + n - 1$ items to be arranged. There are $n(N_i + n - 1)!$ ways to arrange the particles and compartments, but we have overcounted because there are $n!$ permutations of compartments in a cell, and the order in which particles are added to the cell is also irrelevant (the factor $N_i!$ we had in the classical case). Thus

$$W = \prod_i \frac{n\,(N_i + n - 1)!}{N_i! n!} = \prod_i \frac{(N_i + n - 1)!}{N_i!\,(n-1)!} \,.$$

Optimizing this, we find

$$\begin{aligned}\delta \ln W =& \delta \sum_i [(n + N_i - 1) \ln (n + N_i - 1) - N_i \ln N_i \\ & - (n-1) \ln (n-1) - \ln \alpha N_i - \beta w_i N_i] \\ =& \sum_i \left[\ln \frac{n + N_i - 1}{N_i} - \ln \alpha - \beta w_i\right] \delta N_i = 0 \,,\end{aligned}$$

or

$$N_i = (n-1) \left(\alpha e^{w_i/kT} - 1\right)^{-1} \,.$$

In fact, this is the relevant expression when there is no limit to the number of particles that can be put into the compartment of size h^3, *i.e.*, for bosons. Further, in the case when bosons are photons, the condition δN does not apply, and the factor $\alpha \equiv 1$.

For fermions, only 2 particles can be put into a compartment, where 2 is the spin degeneracy. Thus, phase space is composed of $2n$ half-compartments, either full or empty. There are no more than $2n$ things to be arranged and therefore no more than $2n!$ microstates. But again, we overcounted. For N_i filled compartments, the number of indistinguishable permutations is $N_i!$, and the number of indistinguishable permutations of the $2n - N_i$ empty compartments is $(2n - N_i)!$. In this case, we therefore have

$$W = \prod_i \frac{(2n)!}{N_i!\,(2n - N_i)!} \,.$$

As before, we optimize:

$$\begin{aligned}\delta \ln W =& \delta \sum_i [2n \ln (2n) - (2n - N_i) \ln (2n - N_i) \\ & - \ln \alpha N_i - \beta w_i N_i] \\ =& \sum_i \left[\ln \frac{2n - N_i}{N_i} - \ln \alpha - \beta w_i \right] \delta N_i = 0 \,,\end{aligned}$$

or

$$N_i = 2n \left(\alpha e^{w_i/kT} + 1 \right)^{-1} \,.$$

The quantity $\ln \alpha$ can be associated with the negative of the degeneracy parameter μ/T, where μ is the chemical potential, of the system. The classical case is the limit of the fermion or boson case when $\alpha \to \infty$, since in this case the ± 1 in the denominator of the distribution function does not matter. In the boson case, also, $\alpha \geq 1$ since $w_i > 0$ and $N_i > 0$. Bosons become degenerate when $\alpha \to 1$. For photons, $\alpha = 1$. In the fermion case, there is no restriction on the value of α, and fermions become degenerate when $\alpha \to -\infty$.

Thermodynamics

The internal energy U is

$$U = TS - PV + \sum_i \mu_i N_i$$

and the first law is

$$dU = TdS - PdV + \sum_i \mu_i dN_i\,.$$

This implies

$$VdP - SdT - \sum_i N_i d\mu_i = 0\,.$$

The Helmholtz F and Gibbs G free energies are

$$F = U - TS\,; \qquad G = \sum_i \mu_i N_i\,.$$

$$dF = -SdT - PdV + \sum_i \mu_i dN_i\,; \quad dG = VdP - SdT + \sum_i dN_i\,.$$

The thermodynamic potential $\Omega = -PV$ obeys

$$d\Omega = -SdT - PdV - \sum_i N_i d\mu_i\,.$$

The following are useful thermodynamic relations:

$$\left.\frac{\partial U}{\partial S}\right|_{V,N_i} = T \qquad \left.\frac{\partial U}{\partial V}\right|_{S,N_i} = -P \qquad \left.\frac{\partial U}{\partial N_i}\right|_{S,V,N_{j\neq i}} = \mu_i$$

$$\left.\frac{\partial F}{\partial T}\right|_{V,N_i} = -S \qquad \left.\frac{\partial F}{\partial V}\right|_{T,N_i} = -P \qquad \left.\frac{\partial F}{\partial N_i}\right|_{T,V,N_{j\neq i}} = \mu_i$$

$$\left.\frac{\partial \Omega}{\partial T}\right|_{V,\mu_i} = -S \qquad \left.\frac{\partial \Omega}{\partial V}\right|_{T,\mu_i} = -P \qquad \left.\frac{\partial \Omega}{\partial \mu_i}\right|_{T,V,\mu_{j\neq i}} = -N_i$$

Then $\partial P/\partial T|_{V,\mu_i} = S/V$ and $\partial P/\partial \mu_i|_{T,V,\mu_{j\neq i}} = N_i/V$.

Statistical Physics of Perfect Gases—Fermions

The energy of a non-interacting particle is related to its rest mass m and momentum p by the relativistic relation

$$E^2 = m^2c^4 + p^2c^2. \tag{1}$$

The occupation index is the probability that a given momentum state will be occupied:

$$f = \left[1 + \exp\left(\frac{E-\mu}{T}\right)\right]^{-1} \tag{2}$$

for fermions, where $\mu = \partial\epsilon/\partial n|_s$ is the chemical potential and ϵ is the energy density.

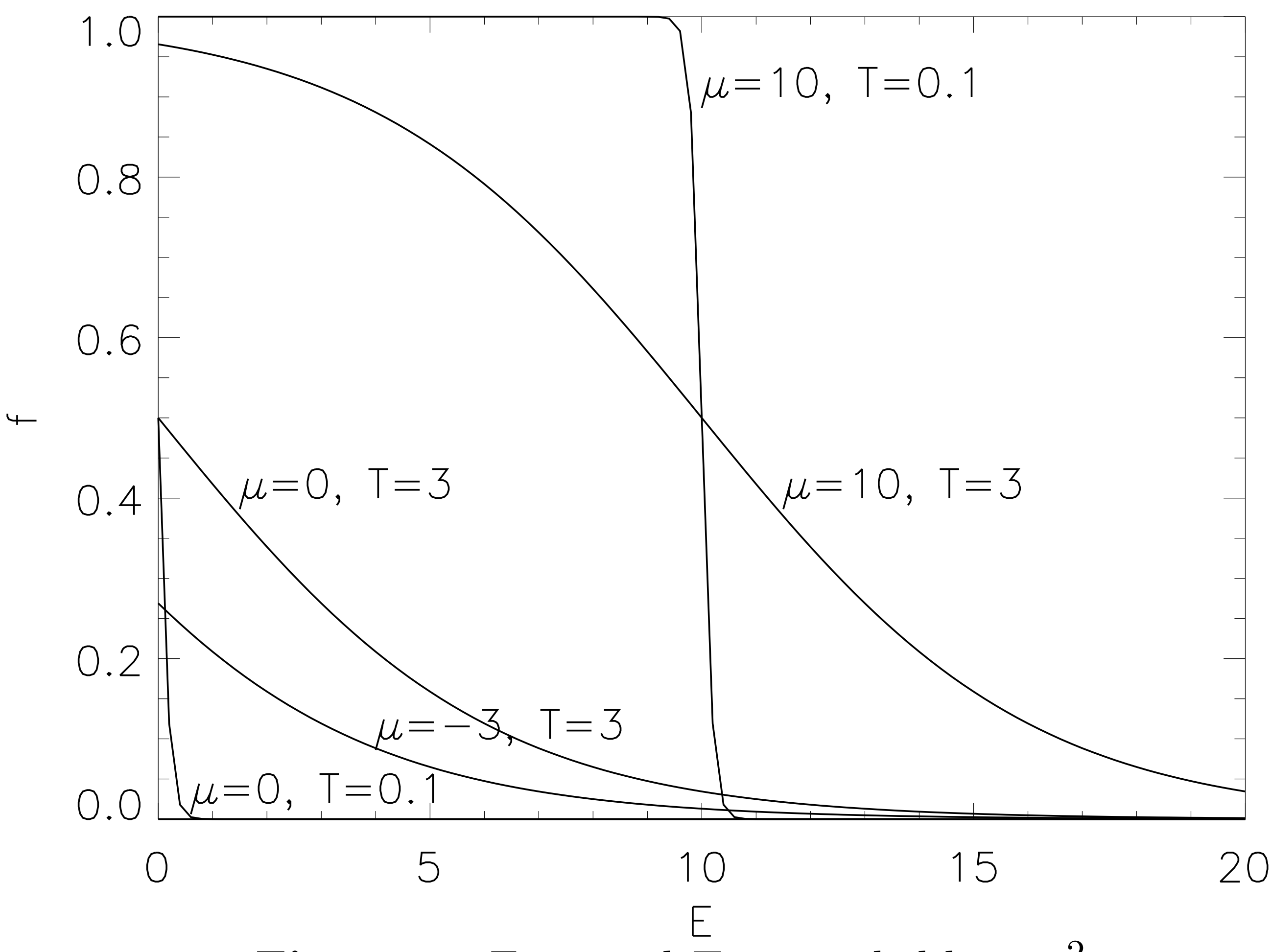


Figure 1: E, μ and T are scaled by mc^2.

When the particles are interacting, E generally contains an effective mass and a potential contribution. μ corresponds to the energy change when 1 particle is added to or subtracted from the system. The entropy per particle is s. We will use units such that k_B=1; thus $T = 1$ MeV corresponds to $T = 1.16 \times 10^{10}$ K. The number and internal energy densities are given, respectively, by

$$n = \frac{g}{h^3} \int f d^3p; \qquad \epsilon = \frac{g}{h^3} \int E f d^3p \tag{3}$$

where g is the spin degeneracy ($g = 2j+1$ for massive particles, where j is the spin of the particle, i.e., $g = 2$ for electrons, muons and nucleons, $g = 1$ for neutrinos). The entropy can be expressed as

$$ns = -\frac{g}{h^3} \int \left[f \ln f + (1-f) \ln(1-f) \right] d^3p \tag{4}$$

and the thermodynamic relations

$$P = n^2 \frac{\partial(\epsilon/n)}{\partial n}|_s = Tsn + \mu n - \epsilon \tag{5}$$

gives the pressure. Incidentally, the two expressions (Eqs. (4) and (5)) are generally valid for interacting gases, also. We also note, for future reference, that

$$P = \frac{g}{3h^3} \int p \frac{\partial E}{\partial p} f d^3p. \tag{6}$$

Thermodynamics gives also that

$$n = \left.\frac{\partial P}{\partial \mu}\right|_T; \qquad ns = \left.\frac{\partial P}{\partial T}\right|_\mu. \tag{7}$$

Note that if we define degeneracy parameters $\phi = \mu/T$ and $\psi = (\mu - mc^2)/T$ the following relations are valid:

$$P = -\epsilon + n \left.\frac{\partial \epsilon}{\partial n}\right|_T + T \frac{\partial P}{\partial T}|_n; \quad \left.\frac{\partial P}{\partial T}\right|_\phi = ns + n\phi; \quad \left.\frac{\partial P}{\partial T}\right|_\psi = ns + n\psi. \tag{8}$$

In many cases, one or the other of the following limits may be realized: extremely degenerate ($\phi \to +\infty$), nondegenerate ($\phi \to -\infty$), extremely relativistic ($p >> mc$), non-relativistic ($p << mc$).

Non-relativistic

In this case, one expands Eq. (3) in the limit $p << mc$. Defining $x = p^2/(2mT)$ and $\psi = (\mu - mc^2)/T$, one has

$$n = \frac{g\,(2mT)^{3/2}}{4\pi^2\hbar^3}\int_0^\infty \frac{x^{1/2}dx}{1+e^{x-\psi}} \equiv \frac{g\,(2mT)^{3/2}}{4\pi^2\hbar^3}F_{1/2}\,(\psi) \qquad (9)$$

$$\epsilon = nmc^2 + \frac{gT\,(2mT)^{3/2}}{4\pi^2\hbar^3}F_{3/2}\,(\psi)\,. \qquad (10)$$

Here, F_i is the usual Fermi integral which satisfies the recursion

$$\frac{dF_i\,(\psi)}{d\psi} = iF_{i-1}\,(\psi)\,. \qquad (11)$$

$$P = \frac{2}{3}\left(\epsilon - nmc^2\right); \qquad\qquad s = \frac{5F_{3/2}\,(\psi)}{3F_{1/2}\,(\psi)} - \psi. \qquad (12)$$

Fermi integrals for zero argument satisfy

$$F_i\,(0) = \left(1 - 2^{-i}\right)\Gamma\,(i+1)\,\zeta\,(i+1)\,, \qquad (13)$$

where ζ is the Riemann zeta function. Note that $F_i(0) \xrightarrow[i\to\infty]{} i!$. $F_i(\psi)$ may be expanded around $\psi = 0$ with

$$F_i\,(\psi) = F_i\,(0) + iF_{i-1}\,(0)\,\psi + \frac{i\,(i-1)}{2}F_{i-2}\,(0)\,\psi^2 + \cdots. \qquad (14)$$

Since $F_0(\psi) = \ln(1+e^{\psi})$, Fermi integrals with integer indices less than 0 do not exist. The recursion Eq. (11) can be employed to define non-integer negative indices, however.

i		$F_i(0)$	i		$F_i(0)$
-7/2		0.249109	3/2		1.152804
-5/2		0.2804865	2		1.803085
-3/2		-1.347436	5/2		3.082586
-1/2		1.07215	3	$7\pi^4/120$	5.682197
0	ln(2)	0.693147	4		23.33087
1/2		.678094	5	$31\pi^6/252$	118.2661
1	$\pi^2/12$	0.822467			

a. **Non-degenerate and non-relativistic**: In this limit, using the expansion

$$F_i(\psi) = \Gamma(i+1) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{n\psi}}{n^{i+1}}, \qquad \psi \to -\infty \tag{15}$$

we find

$$n = g \left(\frac{mT}{2\pi\hbar^2} \right)^{3/2} e^{\psi}, \qquad P = nT, \qquad s = 5/2 - \psi. \tag{16}$$

b. **Degenerate, non-relativistic**: In this limit, we use the Sommerfeld expansion

$$F_i(\psi) = \frac{\psi^{i+1}}{i+1} \sum_{n=0}^{\infty} \frac{(i+1)!}{(i+1-2n)!} \left(\frac{\pi}{\psi} \right)^{2n} C_n, \qquad \psi \to \infty \tag{17}$$

Some values for the constants C_n are $C_0 = 1, C_1 = 1/6, C_2 = 7/360$, and $C_3 = 31/15120$. We find

$$\begin{aligned} n =& \frac{g\left(2m\psi T\right)^{3/2}}{6\pi^2\hbar^3}\left[1+\frac{1}{8}\left(\frac{\pi}{\psi}\right)^2+\cdots\right], \\ P =& \frac{2n\psi T}{5}\left[1+\frac{5}{12}\left(\frac{\pi}{\psi}\right)^2+\cdots\right], \\ s =& \frac{\pi^2}{2\psi}+\cdots. \end{aligned} \tag{18}$$

Extremely relativistic

This case corresponds to setting the rest mass to zero. Eqs. (3) and (5) become

$$\begin{aligned} n =& \frac{g}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 F_2\left(\phi\right), \\ P =& \frac{\epsilon}{3} = \frac{gT}{6\pi^2}\left(\frac{T}{\hbar c}\right)^3 F_3\left(\phi\right), \\ s =& \frac{4F_3\left(\phi\right)}{3F_2\left(\phi\right)} - \phi. \end{aligned} \tag{19}$$

The above limiting expressions for the Fermi integrals may be used in these expressions.

a. **Extremely relativistic and non-degenerate**: Use of the expansion Eq. (15) results in

$$\begin{aligned} n =& \frac{g}{\pi^2}\left(\frac{T}{\hbar c}\right)^3 e^{\phi}, \\ P =& nT, \qquad s = 4 - \ln\left[\frac{\pi^2 n}{g}\left(\frac{\hbar c}{T}\right)^3\right] = 4-\phi. \end{aligned} \tag{20}$$

b. **Extremely relativistic and extremely degenerate**: The expansion Eq. (17) gives

$$
\begin{aligned}
n =& \frac{g}{6\pi^2}\left(\frac{\mu}{\hbar c}\right)^3\left[1+\left(\frac{\pi}{\phi}\right)^2+\cdots\right],\\
P =& \frac{n\mu}{4}\left[1+\left(\frac{\pi}{\phi}\right)^2+\cdots\right],\\
s =& \frac{\pi^2}{\phi}+\cdots
\end{aligned}
\tag{21}
$$

Extremely degenerate

This case corresponds to $\phi >> 0$. It is useful to define the Fermi momentum p_f for which the occupation index $f = 1/2$, i.e., where $\mu = E_f = \sqrt{m^2c^4 + p_f^2c^2}$. In terms of the parameter $x = p_f/mc$, we have

$$\mu = mc^2\sqrt{1+x^2}. \tag{22}$$

In the case $\phi \to \infty$, Eq. (2) becomes a step function, with $f = 1$ for $E \leq \mu$; $f = 0$ for $E > \mu$.

$$
\begin{aligned}
n =& \frac{8A}{mc^2}x^3,\\
P =& A\left[x\left(2x^2-3\right)\sqrt{1+x^2}+3\sinh^{-1}x\right],\\
\epsilon - nmc^2 =& A\left[3x\left(2x^2+1\right)\sqrt{1+x^2}-8x^3-3\sinh^{-1}x\right],\\
s =& 0,
\end{aligned}
\tag{23}
$$

where $A = (gmc^2/48\pi^2)(mc/\hbar)^3$.

Non-degenerate

This case corresonds to $\phi << 0$. Because pair creation is often important in this case, we delay detailed discussion of limiting formulae for a later section. If pairs are neglected, results may be expressed in terms of Bessel functions:

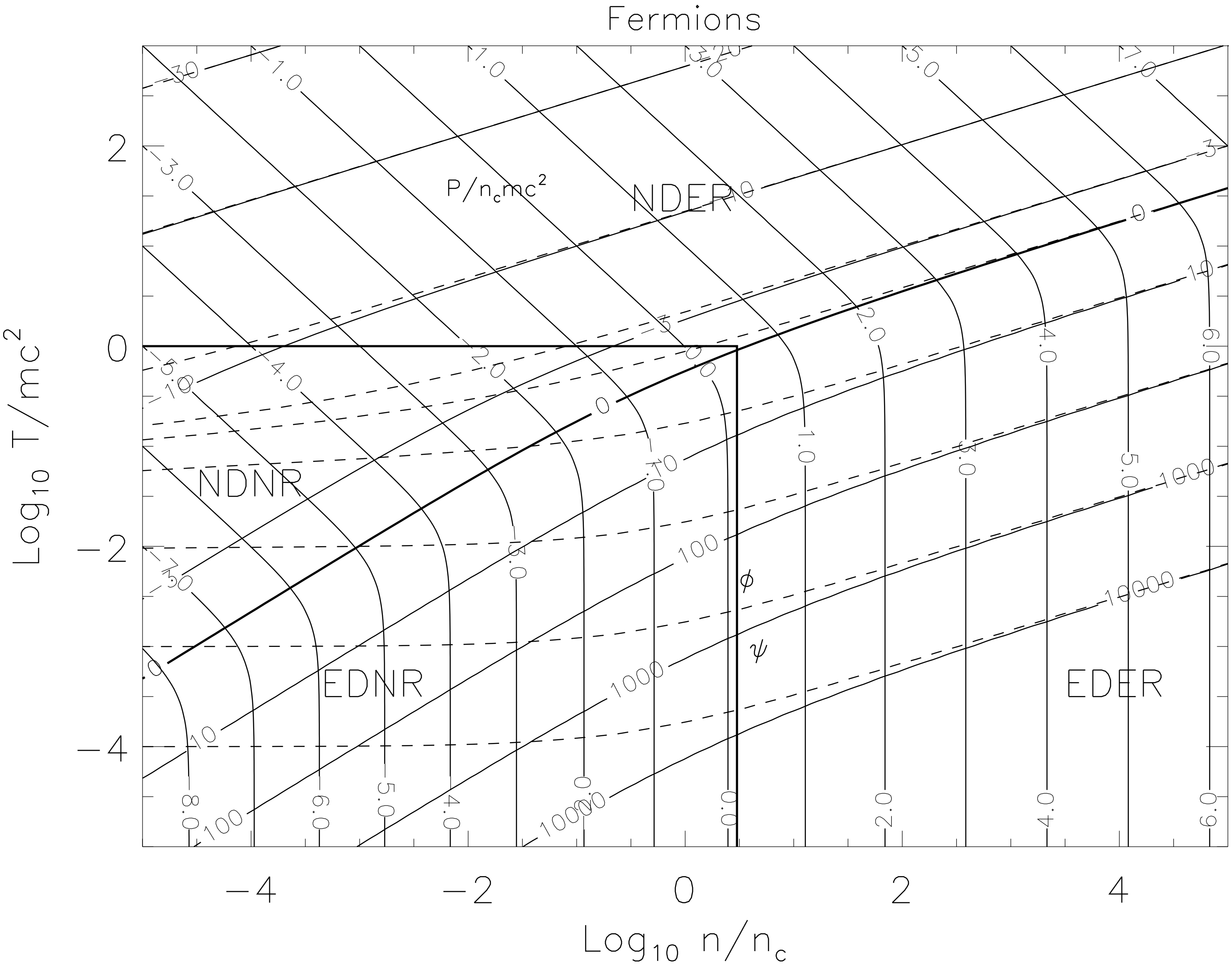
$$\begin{aligned} n &= \left(\frac{mc}{\hbar}\right)^3 \frac{T}{3mc^2} e^{\phi} K_2\left(mc^2/T\right), \\ P &= nT, \\ \epsilon - nmc^2 &= \left(\frac{mc}{\hbar}\right)^3 \frac{T}{3} e^{\phi} [-K_1\left(mc^2/T\right) + \\ &\quad \left(3T/mc^2 - 1\right) K_2\left(mc^2/T\right)], \\ s &= 4 - \frac{mc^2}{T}\frac{K_1\left(mc^2/T\right)}{K_2\left(mc^2/T\right)} - \phi. \end{aligned} \tag{24}$$

General Comments About Fermions

Convenient scalings for electrons are achieved using

$$n_c = \left(\frac{g}{2\pi^2}\right)\left(\frac{m_e c}{\hbar}\right)^3 = 1.76 \times 10^{-9}\ \mathrm{fm}^{-3}$$

$$m_e c^2 = 0.511\ \mathrm{MeV}$$



Fermions become relativistic under non-degenerate conditions when $T > mc^2$ ($T > 5 \times 10^9$ K for electrons) for any density, and, under degenerate conditions, when $p_f c > mc^2$ ($\rho Y_e > 2 \times 10^6$ g cm^{-3} for electrons) for any temperature. Here, ρ is the baryon density, and the number of electrons per baryon is Y_e. $n(\equiv n_e) = \rho N_o Y_e$. N_o is Avogadro's number.

$\psi \simeq 0$ demarks the degenerate and non-degenerate regions under all relativity conditions.

$$n = g\frac{(2mT)^{3/2}}{4\pi^2\hbar^3}F_{1/2}(0);$$

$$\rho Y_e \simeq 2 \times 10^6 \left(\frac{T}{5\times 10^9 \text{ K}}\right)^{3/2} \text{ g cm}^{-3} \quad \text{non}-\text{relativistic};$$

$$n = \frac{g}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 F_2(0);$$

$$\rho Y_e \simeq 2 \times 10^6 \left(\frac{T}{5\times 10^9 \text{ K}}\right)^{3} \text{ g cm}^{-3} \qquad \text{relativistic}$$

separate the degenerate from the non-degenerate regions.

Interacting baryons are far more complicated. At subnuclear densities ($\rho < \rho_o \equiv 2.7 \times 10^{14}$ g cm^{-3}) they cluster into nuclei with internal densities near ρ_o. The nuclei themselves are dilute, comprising a non-degenerate, non-relativistic gas, but with a strong Coulombic (lattice) interaction. At very high temperatures, the nuclei dissociate. Above ρ_o, nuclear interactions and degeneracy effects dominate. Baryons become relativistic at a density $(m_{baryon}/m_{electron})^3$ times higher than the electrons, or about 10^{16} g cm^{-3}. This is above the transition density to quark matter. At these densities, quarks can be approximated as a perfect gas due to asymptotic freedom.

Fermion–Antifermion particle pairs

Under conditions found in the evolution of very massive stars, the temperature may be high enough to produce electron-positron pairs, while the electrons are non-relativistic. During gravitational collapse a degenerate neutrino-antineutrino gas forms when densities large enough to trap neutrinos on dynamical time scales are reached ($\rho > 10^{12}$ g/cm^3). For particle-antiparticle pairs in equilibrium, $\mu_+ = -\mu_-$. The *net difference* of particles and anti-particles and the *total* pressure are

$$\begin{aligned} n =& n_+ - n_- = \frac{4\pi g}{h^3}\int_0^\infty p^2 \left[\frac{1}{1+e^{(E-\mu)/T}} - \frac{1}{1+e^{(E+\mu)/T}}\right] dp, \\ P =& P_+ + P_- = \frac{4\pi g}{3h^3}\int p^3 \frac{\partial E}{\partial p}\left[\frac{1}{1+e^{(E-\mu)/T}} + \frac{1}{1+e^{(E+\mu)/T}}\right] dp. \end{aligned} \tag{25}$$

Thus, when pairs are included, and n is positive, $\mu \equiv \mu_+$ must be positive, i.e., there will not be cases involving extreme non-degeneracy. However, pairs will never be important whenever $\mu/T >> 0$, that is, under extremely degenerate conditions. With the substitutions $x = pc/T$, $z = mc^2/T$, we may write

$$\begin{aligned} n =& \frac{g}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 \sinh\phi \int_0^\infty \frac{x^2}{\cosh\phi + \cosh\sqrt{z^2+x^2}} dx, \\ P =& \frac{gT}{6\pi^2}\left(\frac{T}{\hbar c}\right)^3 \int_0^\infty \frac{x^4}{\sqrt{z^2+x^2}}\left[\frac{\cosh\phi + e^{-\sqrt{z^2+x^2}}}{\cosh\phi + \cosh\sqrt{z^2+x^2}}\right] dx. \end{aligned} \tag{26}$$

a. Extremely relativistic case: $\mu >> mc^2$ or $T >> mc^2$. This applies to neutrinos. With $\mu = \mu_+ = -\mu_-$, i.e., $z \to 0$,

$$n = n_+ - n_- = \frac{g}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 \left[F_2\left(\frac{\mu}{T}\right) - F_2\left(-\frac{\mu}{T}\right)\right]$$
$$= \frac{g}{6\pi^2}\left(\frac{\mu}{\hbar c}\right)^3 \left[1 + \left(\frac{\pi T}{\mu}\right)^2\right];$$
$$\epsilon/3 = P = P_+ + P_- = \frac{gT}{6\pi^2}\left(\frac{T}{\hbar c}\right)^3 \left[F_3\left(\frac{\mu}{T}\right) + F_3\left(-\frac{\mu}{T}\right)\right]$$
$$= \frac{g\mu}{24\pi^2}\left(\frac{\mu}{\hbar c}\right)^3 \left[1 + 2\left(\frac{\pi T}{\mu}\right)^2 + \frac{7}{15}\left(\frac{\pi T}{\mu}\right)^4\right];$$
$$s = \frac{gT\mu^2}{6n\left(\hbar c\right)^3}\left[1 + \frac{7}{15}\left(\frac{\pi T}{\mu}\right)^2\right]. \tag{27}$$

These expressions are *exact*. The exponential terms ignored in the Sommerfeld expansion of the $+\mu/T$ Fermi integral are exactly canceled by those of the $-\mu/T$ Fermi integral. The pair Fermi integral

$$G_i\left(\eta\right) \equiv F_i\left(\eta\right) + \left(-1\right)^{i+1} F_i\left(-\eta\right) \qquad i \geq 0$$

obeys the same recursion formula as $F_i(\eta)$ for $i \geq 1$.

$n(\mu)$ is a cubic in μ, which can be inverted:

$$\mu = r - q/r, \qquad r = \left[\left(q^3 + t^2\right)^{1/2} + t\right]^{1/3}, \tag{28}$$

where $t = 3\pi^2(\hbar c)^3 n/g$ and $q = (\pi T)^2/3$. For $T \to \infty$, one has $\mu \to 6n(\hbar c)^3/gT^2 \to 0^+$. For all μ and T the adiabatic index

$$\Gamma_1 = \frac{d\ln P}{d\ln n}\Big|_s = \frac{d\ln P}{d\ln n}\Big|_T + \frac{T}{P}\left(\frac{dP}{dT}\right)_n^2 \left(\frac{d\epsilon}{dT}\right)_n^{-1} = 4/3. \tag{29}$$

One may include the lowest order corrections for finite rest mass by expanding the integrands of Eq. (3) and using the recursion relations for the Fermi integrals:

$$
\begin{aligned}
n &= \frac{g}{6\pi^2}\left(\frac{\mu}{\hbar c}\right)^3\left[1+\mu^{-2}\left(\pi^2T^2-\frac{3}{2}m^2c^4\right)\right],\\
P &= \frac{g\mu}{24\pi^2}\left(\frac{\mu}{\hbar c}\right)^3\left[1+\mu^{-2}\left(2\pi^2T^2-3m^2c^4\right)+\frac{\pi^2T^2}{\mu^4}\left(\frac{7}{15}\pi^2T^2-m^2c^4\right)\right],\\
\epsilon &= \frac{g\mu}{8\pi^2}\left(\frac{\mu}{\hbar c}\right)^3\left[1+\mu^{-2}\left(2\pi^2T^2-m^2c^4\right)+\frac{\pi^2T^2}{\mu^4}\left(\frac{7}{15}\pi^2T^2-\frac{1}{3}m^2c^4\right)\right],\\
s &= \frac{gT\mu^2}{6n\left(\hbar c\right)^3}\left[1+\mu^{-2}\left(\frac{7}{15}\pi^2T^2-\frac{1}{2}m^2c^4\right)\right].
\end{aligned}
\tag{30}
$$

The relativistic relationship $\epsilon = 3P$ no longer holds. Interestingly, the cubic relationship between μ and n is preserved in this approximation, and the solution is still given by Eq. (28) if we simply redefine $q = (\pi T)^2/3 - m^2c^4/2$. Including the finite rest mass terms lowers Γ_1 below 4/3:

$$\Gamma_1 = \frac{4}{3}\left(1-\frac{5}{11}\left(\frac{mc^2}{\pi T}\right)^2\right) \tag{31}$$

when photons (see below) are also included.

b. Non-relativistic case: $\mu << mc^2$ and $T << mc^2$.

In the degenerate case, $T \to 0$, $\mu \to (mc^2)^+$ and pairs are of negligible importance. We can use the non-relativistic, degenerate formulas already obtained for particles alone. At higher temperatures, μ reaches a maximum, and then decreases, eventually becoming less than mc^2, so that $\mu' < 0$. The gas is thus at most only partially degenerate when pairs are present and

$n = n_+ - n_- > 0$. The non-degenerate expansion yields

$$n_\pm \simeq g \left(\frac{mT}{2\pi\hbar^2} \right)^{3/2} e^{[\pm\mu - mc^2]/T}. \tag{32}$$

Noting that $n = n_+ - n_-$ and

$$n_+ n_- = g^2 \left(\frac{mT}{2\pi\hbar^2} \right)^{3} e^{-2mc^2/T} \equiv n_1^2, \tag{33}$$

we can instead write

$$n_\pm = \mp \frac{n}{2} + \left[\left(\frac{n}{2} \right)^2 + n_1^2 \right]^{1/2}. \tag{34}$$

$$P = (n_+ + n_-)\, T = \left(n^2 + 4 n_1^2 \right)^{1/2} T, \tag{35a}$$

$$\epsilon = (n_+ + n_-) \left(mc^2 + \frac{3}{2} T \right), \tag{35b}$$

$$s = \left(\frac{5}{2} + \frac{mc^2}{T} \right) \frac{(n_+ + n_-)}{n} - \frac{\mu}{T}, \tag{35c}$$

$$\mu = T \ln \left[\frac{n}{2n_1} + \left(\frac{n^2}{4n_1^2} + 1 \right)^{1/2} \right]. \tag{35d}$$

Pairs are important in the non-relativistic case when $n \leq n_1$. Including photon pressure (see below), in the case when $n << n_1$, one has

$$\Gamma_1 \simeq \frac{4}{3} \left[1 - \frac{15}{32} \left(\frac{2mc^2}{\pi T} \right)^{7/2} e^{-mc^2/T} \right]. \tag{36}$$

Thus Γ_1 reaches a minimum value (1.02) when $T = \frac{2}{7} mc^2$, and is always less than $\frac{4}{3}$. The creation of a pair costs an energy of $2mc^2$ which is non-negligible in the non-relativistic case.

c. Non-degenerate case: $\psi < 0$

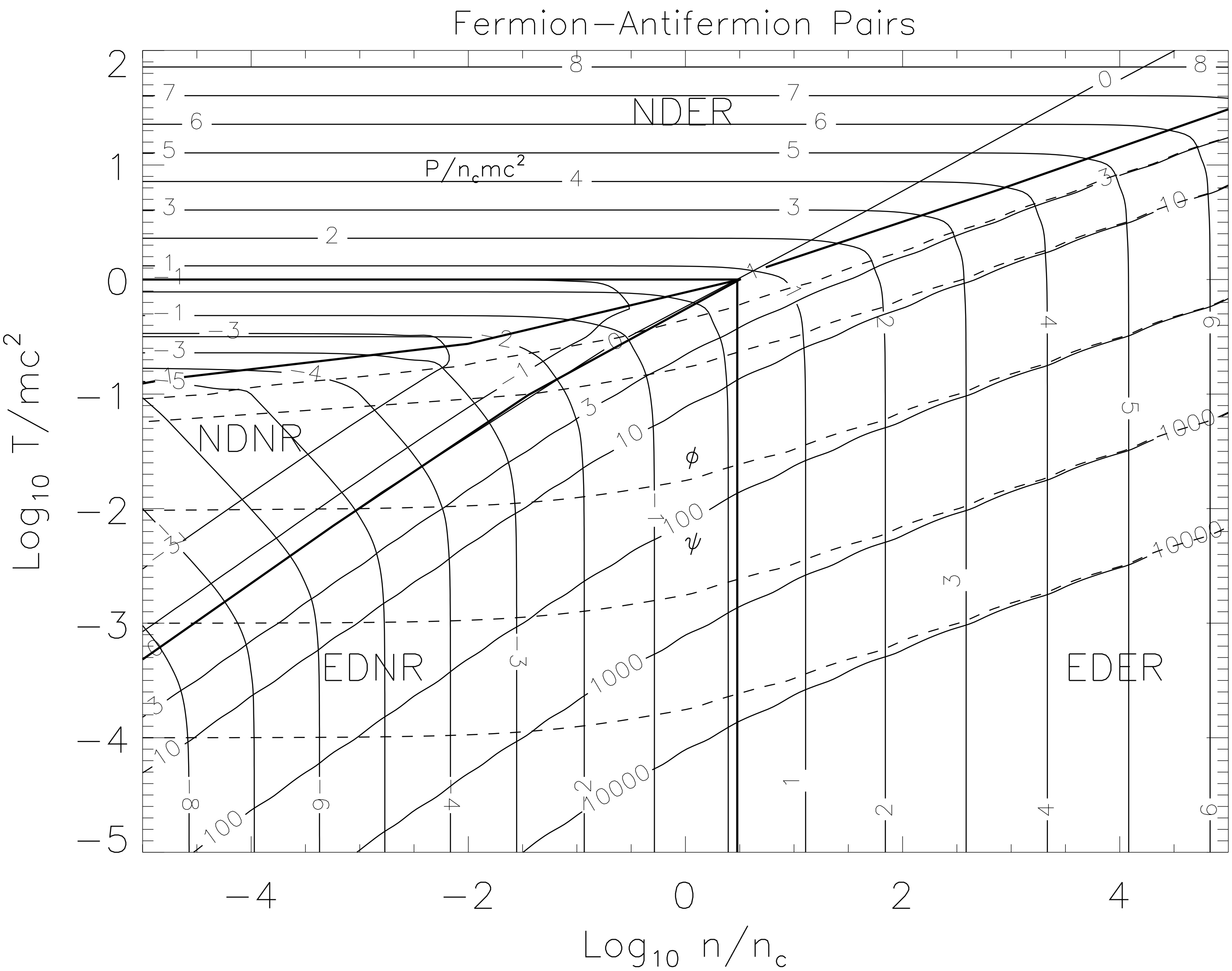
Consider the region for which $\cosh\phi - 1 << \cosh z$. Since $\mu = T\phi$ cannot be negative when pairs are included, the gas is at most partially degenerate in this non-degenerate limit. Expanding the $\cosh\phi$ in the denominator terms to lowest order in $\cosh\phi - \cosh z$,

$$n = n_c z^{-3} \sinh\phi \int_0^\infty \frac{x^2 dx}{1 + \cosh\sqrt{z^2 + x^2}},$$
$$P = n_c T z^{-3} [(\cosh\phi - 1) \int_0^\infty \frac{x^2 dx}{1 + \cosh\sqrt{z^2 + x^2}}$$
$$+ \frac{2}{3} \int_0^\infty \frac{x^4 dx}{\sqrt{x^2 + z^2}} \frac{1}{1 + e^{\sqrt{x^2+z^2}}}], \tag{37}$$
$$\epsilon = n_c T z^{-3} [(\cosh\phi - 1) \int_0^\infty \frac{x\left(x^2 + z^2\right) dx}{1 + \cosh\sqrt{z^2 + x^2}}$$
$$+ 2 \int_0^\infty \frac{x^2\sqrt{z^2 + x^2}}{1 + e^{\sqrt{z^2+x^2}}} dx],$$

where $n_c = (g/2\pi^2)(mc/\hbar)^3 = 6 \times 10^6$ g cm^{-3} for $g = 2$. This is an interesting approximation because, given n and T, one can immediately evaluate μ or ϕ because they no longer appear within the integrals. The integrals can be easily evaluated by quadrature, with relatively few points, using Gauss-Laguerre for $z < 30$ and Gauss-Hermite for $z > 30$.

When are pairs important?

In the relativistic case, $n_- = 0.1 n_+$ is equivalent to $F_2(\phi) \simeq 10F_2(-\phi)$ or $\phi \simeq 0.9$. In the non-relativistic case, we find $\phi \simeq \ln\sqrt{10}$ or $\phi \simeq 1.15$. $\phi \simeq 1$ is the effective boundary. The intrusion of this boundary into the NDNR region means that there are actually five limiting cases when pairs are considered, as opposed to four when pairs are ignored. This is an unfortunate complication.



Generalized Approximation

We explore here a technique invented by Eggleton, Faulkner and Flannery (*A& A* **23**, 325 [1973]) to bridge the limiting regions for a fermion gas. It is essential to maintain thermodynamic consistency in this approximation. To include pairs we simply apply the scheme separately to electrons and positrons. The scheme establishes an analytic formula for the thermodynamic potential (or pressure) as an explicit function of chemical potential and temperature. Then $n = T^{-1}\partial P/\partial\psi; ns =$

$\partial P/\partial T - n\psi; \epsilon = T(\partial P/\partial T) - P + nmc^2$. Density and temperature are inputs, so iteration is necessary to determine the chemical potential. Johns, Ellis & Lattimer (*ApJ* **473**, 1020 [1996]) improved the accuracy of the scheme and corrected the behavior of the entropy in the degenerate limit.

The four limiting cases we have discussed are:

$$\frac{P}{n_c mc^2} = \begin{cases} (\psi T)^4 \sum\sum a_{mn}\psi^{-2m}(\psi T)^{-n} & \mathrm{ER,ED}: \psi T >> mc^2, \psi >> 1 \\ (\psi T)^{5/2} \sum\sum b_{mn}\psi^{-2m}(\psi T)^{n} & \mathrm{NR,ED}: \psi T << mc^2, \psi >> 1 \\ T^4 e^{\psi} \sum\sum c_{mn} e^{m\psi} T^{-n} & \mathrm{ER,ND}: \psi << -1, T >> mc^2 \\ T^{5/2} e^{\psi} \sum\sum d_{mn} e^{m\psi} T^{n} & \mathrm{NR,ND}: \psi << -1, T << mc^2 \end{cases} \tag{38}$$

where $n_c = (g/2\pi^2)(mc/\hbar)^3$. The coefficients a_{mn}, b_{mn}, c_{mn} and d_{mn} $(m, n \in 0\ldots\infty)$ can be determined from the limits. The key is to find functions $f(\psi), g(\psi, T)$ such that Eq. (38) can be rewritten as

$$\frac{P}{n_c mc^2} = \begin{cases} g^4 \sum\sum a'_{mn} f^{-m} g^{-n} & \mathrm{ER,ED}: g >> 1, \quad f >> 1 \\ g^{5/2} \sum\sum b'_{mn} f^{-m} g^{n} & \mathrm{NR,ED}: g << 1, \quad f >> 1 \\ f g^4 \sum\sum c'_{mn} f^{m} g^{-n} & \mathrm{ER,ND}: g >> 1, \quad f << 1 \\ f g^{5/2} \sum\sum d'_{mn} f^{m} g^{n} & \mathrm{NR,\ ND}: g << 1, \quad f << 1. \end{cases} \tag{39}$$

This is possible provided that

$$f(\psi) = \begin{cases} \psi^2 \sum r_m \psi^{-2m} & \mathrm{ED}: \quad \psi >> 1 \\ e^{\psi} \sum s_m e^{m\psi} & \mathrm{ND}: \quad \psi << -1 \end{cases} \tag{40}$$

and

$$g(\psi, T) = \begin{cases} \psi T \sum t_m \psi^{-2m} & \mathrm{ED}: \quad \psi >> 1 \\ T \sum u_m e^{m\psi}. & \mathrm{ND}: \quad \psi << -1 \end{cases} \tag{41}$$

r_{mn}, s_{mn}, t_{mn} and u_{mn} are additional coefficients. Then

$$\frac{P}{n_c mc^2} = \frac{f}{1+f} g^{5/2} (1+g)^{3/2} \frac{\sum_0^M \sum_0^N P_{mn} f^m g^n}{(1+f)^M (1+g)^N} \tag{42}$$

has the proper limits for any $M, N \geq 1$ when f and g are either large or small. P_{mn} are coefficients which are least squares fit to a numerical evaluation of P. From Eq. (41), $g \propto T$, and

$$g = \left(T/mc^2\right) \sqrt{1+f} \tag{43}$$

guarantees the right limiting behavior in Eq. (41).

On the other hand, it is also clear that $\partial f/\partial\psi$ is either $\sqrt{f}$ $(f \to \infty)$ or f $(f \to 0)$. We choose

$$\partial f/\partial\psi = f/\sqrt{1+f/a}, \tag{44}$$

where a is an adjustable parameter introduced by Johns, Ellis & Lattimer (1996) which greatly improves the accuracy of the scheme. Integrating this relation, we find the required relation between ψ and f:

$$\psi = 2\sqrt{1+f/a} + \ln \frac{\sqrt{1+f/a}-1}{\sqrt{1+f/a}+1}. \tag{45}$$

Explicitly evaluating the density, we find

$$n = \frac{1}{T}\frac{\partial P}{\partial\psi}\Big|_T = \frac{1}{T}\frac{\partial f}{\partial\psi}\left(\frac{\partial P}{\partial f}\Big|_g + \frac{\partial g}{\partial f}\Big|_T \frac{\partial P}{\partial g}\Big|_f\right); \tag{46}$$

$$\frac{n}{n_c} = \frac{f g^{3/2} \sum_0^M \sum_0^N P_{mn} f^m g^n}{\sqrt{1+f/a}\,(1+f)^{M+1/2}(1+g)^{N-3/2}} \times \left[1 + m + \frac{f}{1+f}\left(\frac{1}{4} + \frac{n}{2} - M\right) + \frac{fg}{(1+f)(1+g)}\left(\frac{3}{4} - \frac{N}{2}\right)\right]. \tag{47}$$

From $P+U = T(\partial P/\partial T)_\psi$, where $U = \epsilon - nmc^2$ is the internal energy density,

$$\frac{U}{n_c mc^2} = fg^{5/2}\left(1+g\right)^{3/2} \frac{\sum_0^M \sum_0^N P_{mn} f^m g^n}{(1+f)^{M+1}(1+g)^N} \times \left[\frac{3}{2} + n + \frac{g}{1+g}\left(\frac{3}{2} - N\right)\right]. \tag{48}$$

Given n and T, we invert Eq. (47) to determine f; g is trivially found. The pressure is given by Eq. (42), the chemical potential by Eq. (45), and the energy density by Eq. (48).

The entropy is found from $s = n^{-1}(\partial P/\partial T)_\psi - \psi$. A drawback of the Eggleton et al. scheme was that in the degenerate limit, although the entropy per particle has the correct asymptotic dependence $1/\sqrt{f}$, the coefficient is not exact:

$$s = \begin{cases} \sqrt{\frac{a}{f}}\left(2 + \frac{1}{a} - \frac{M}{a} + \frac{P_{M-1,N}}{aP_{M,N}}\right) \xrightarrow[M,N\to\infty]{} \frac{\pi^2}{2}\sqrt{\frac{a}{f}} & \mathrm{ED, ER} \\ \frac{8}{5}\sqrt{\frac{a}{f}}\left(\frac{5}{4} + \frac{1}{4a} - \frac{M}{a} + \frac{P_{M-1,0}}{aP_{M,0}}\right) \xrightarrow[M,N\to\infty]{} \frac{\pi^2}{4}\sqrt{\frac{a}{f}}. & \mathrm{ED, NR} \end{cases} \tag{49}$$

For $M, N = 2(3)$, Eqs. (49) have errors of 1.35 (0.0165)% and 0.254 (0.0637)%, respectively, for the ER and NR cases. The original Eggleton et al. errors for these cases (they assumed $a = 1$) are (2.4 (1.40)%, 1.0 (0.563)%), respectively.

From Eqs. (49), the corner values of P_{mn} should be:

P_{mn}	$n=0$	$n=N$
$m=0$	$e^2\sqrt{\pi/32}/a$	$e^2/(2a)$
$m=M-1$	$\frac{5\pi^2-40+(32M-8)/a}{15a^{1/4}}$	$\frac{2\pi^2-8+4(M-1)/a}{3a}$
$m=M$	$32/(15a^{5/4})$	$4/(3a^2)$

Johns et al. constrained the fit so that these corner values and the ED entropies in Eq. (49) are exactly fulfilled. For $M=N=3$ we find an optimum fit for $a=0.433$ with a root-mean-square error of $8.1\cdot10^{-5}$ and a maximum error of $3.0\cdot10^{-4}$ at the fitting points. The coefficients P_{mn} are:

P_{mn}	$n=0$	$n=1$	$n=2$	$n=3$
$m=0$	5.34689	18.0517	21.3422	8.53240
$m=1$	16.8441	55.7051	63.6901	24.6213
$m=2$	17.4708	56.3902	62.1319	23.2602
$m=3$	6.07364	18.9992	20.0285	7.11153

To extend this scheme to include pairs, let the subscript + refer to particles and − refer to antiparticles. Then

$$n = n_+\left(f_+, g_+\right) - n_-\left(f_-, g_-\right) \tag{50}$$

where $g_\pm = (T/mc^2)\sqrt{1+f_\pm}$ and

$$\psi_\pm = 2\sqrt{1+f_\pm/a} + \ln\frac{\sqrt{1+f_\pm/a}-1}{\sqrt{1+f_\pm/a}+1}. \tag{51}$$

One solves the simultaneous equations

$$\begin{aligned} A &= n - n_+\left(f_+, T\right) + n_-\left(f_-, T\right) = 0; \\ B &= \psi_+\left(f_+\right) + \psi_-\left(f_-\right) + 2mc^2/T = 0, \end{aligned} \tag{52}$$

where the second follows from $\mu_- = -\mu_+$. This is readily handled, since the derivatives are analytic:

$$\partial A/\partial f_\pm = \mp \partial n_\pm/\partial f_\pm; \qquad \partial B/\partial f_\pm = \sqrt{1 + f_\pm/a}/f_\pm. \quad (53)$$

Boson Gas

The boson pressure and energy density are obtained by employing the same equations as for fermions, but using the Bose distribution function

$$f_B = \left[\exp\left(\frac{E-\mu}{T}\right) - 1\right]^{-1}, \quad (54)$$

and a slightly different entropy formula

$$ns = -\frac{g}{h^3}\int \left[f_B \ln f_B - (1+f_B)\ln(1+f_B)\right] d^3p. \quad (55)$$

Since the occupation index cannot be negative, a free (non-interacting) Bose gas $\mu \le mc^2$. If $\mu \to mc^2$, a "Bose condensate" appears and there will be a finite number of particles in a zero-momentum state. Some limiting cases:

a. Extremely Non-degenerate:

In the non-degenerate limit, $\mu/T \to -\infty$, the Bose and Fermion distributions become indistinguishable, so the limits for thermodynamic quantities evaluated previously for the Fermi gas apply.

b. Extremely Degenerate

For bosons, the "degenerate" limit is $\mu = mc^2$ or $\psi = 0$; the number density is

$$n = \frac{g}{2\pi^2\hbar^3}\int_0^\infty \frac{p^2}{e^{(E-mc^2)/T} - 1}dp. \quad (56)$$

The integrals in this case can be written simply in terms of zero argument Fermi integrals:

$$\int_0^\infty \frac{x^i}{e^x - 1}dx = \left(1 - 2^{-i}\right)^{-1} F_i(0) = \Gamma(i+1)\,\zeta(i+1). \quad (57)$$

Therefore we have the following additional limits:

i. **Relativistic** ($T >> mc^2, \psi = 0$)

$$n = \frac{4g}{6\pi^2}\left(\frac{T}{\hbar c}\right)^3 F_2(0), \qquad \epsilon = 3P;$$

$$P = \frac{4gT}{21\pi^2}\left(\frac{T}{\hbar c}\right)^3 F_3(0) = \frac{g\pi^2 T}{90}\left(\frac{T}{\hbar c}\right)^3, \qquad s = \frac{\pi^4}{15F_2(0)} \simeq 3.601571.$$

ii. **Non-relativistic** ($T << mc^2, \psi = 0$)

$$n = \frac{g}{\pi^2}\frac{(mT)^{3/2}}{\hbar^3}\frac{F_{1/2}(0)}{\sqrt{2}-1}, \qquad \epsilon = nmc^2 + \frac{3}{2}P;$$

$$P = \frac{4gT}{3\pi^2}\frac{(mT)^{3/2}}{\hbar^3}\frac{F_{3/2}(0)}{2^{3/2}-1}, \qquad s = \frac{10}{3}\frac{F_{3/2}(0)}{F_{1/2}(0)}\frac{2^{1/2}-1}{2^{3/2}-1} \simeq 1.283781.$$

Note that in these limits the entropy per boson is constant. The location of the $\psi = 0$ trajectory in a boson density-temperature plot is not far from the same curve for fermions.

c. Extremely Relativistic

In this case, we take $m \to 0$, and we arrive at the simplest bose gas, the photon gas, for which $\mu_\gamma = 0$. With $g_\gamma = 2$, one obtains

$$\epsilon_\gamma = 3P_\gamma = \frac{3}{4}TS_\gamma = \frac{\pi^2 T}{15}\left(\frac{T}{\hbar c}\right)^3, \tag{58}$$

which is $(8/7g)$ times the value for a relativistic fermion gas. Here S is the entropy density. In any regime where electron-positron pairs are important, the photon pressure is also important. In the non-degenerate, relativistic domain, the total pressure from photons and electron-positron pairs is therefore $11P_\gamma/4$, Under situations when neutrino pairs of all three flavors are trapped in the matter, the total pressure increases to $43P_\gamma/8$. In the regime where electrons are degenerate, however, photon pressure is negligible.

In the regime where the electrons are non-degenerate and pairs are not important, the non-degenerate gas pressure of nuclei must be included, and photon pressure may or not be important. The pressure due to photons is important at lower temperatures than pair pressure, owing to the expense of creating electron-positron pairs. Since the photon pressure is $(8/7g)$ times the relativistic non-degenerate pair pressure, the boundary to the region in which photon pressure is significant is simply obtained by a continuation of the straight line relativistic boundary $\rho \propto T^3$ to low densities and is akin to the line $\phi = 1$ in the fermion-antifermion pair case.

Even excluding the contribution from electron-positron pairs, the adiabatic index of a non-relativistic gas changes from 5/3 to 4/3 as the temperature is increased and the contribution of radiation pressure increases. Denoting the fraction of the total pressure due to gas pressure (assuming complete ionization) by β,

$$\Gamma_1 = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta}. \tag{59}$$

This ultimately sets an upper limit to the mass of main sequence stars.

Johns, Ellis & Lattimer (1996) extended the Eggleton et al. scheme to handle a boson gas.