Major Nuclear Burning Stages

Generally, fusion reactions occur with the species of least charge due to the Gamow factor:

$$r_{12} \propto n_1 n_2 \exp\left[-42.48\left(\frac{Z_1^2 A_2^2 A}{T_6}\right)^{1/3}\right].$$

$$p + p \rightarrow \mathrm{He}^2 \rightarrow p + p$$
$$p + \mathrm{He}^4 \rightarrow \mathrm{Li}^5 \rightarrow p + \mathrm{He}^4$$
$$\mathrm{He}^4 + \mathrm{He}^4 \rightarrow \mathrm{Be}^8 \rightarrow \mathrm{He}^4 + \mathrm{He}^4$$

In a p−He gas, the strong nuclear force produces no 2-particle exothermic reactions.

Proton-proton cycle

Hans Bethe first demonstrated that the weak nuclear force allows exothermic fusion reactions, due to the fact that a deuteron is bound by 1.44 MeV relative to two protons.

The weak interaction involved is an allowed decay, but is still weak: the effective cross section is about 10^{-47} cm^2 at 1 MeV laboratory kinetic energy. This is too small to be experimentally confirmed. A thick hydrogen target would have to be bombarded for 10 years with 1 amp of 1-MeV protons to produce a single reaction. The slowness of the reaction allows stars to live long enough for life to form.

The effective S-factor is $S(0) \simeq 3.8\times10^{-22}$ keV-barn, $dS/dE \simeq 4.2 \times 10^{-24}$ barn. The $p-p$ rate without screening is

$$r_{pp} = 3.1 \times 10^{-37} n_p^2 T_6^{-2/3} \exp\left[-33.81 T_6^{-1/3}\right] \times$$
$$\times \left[1 + 0.0123 T_6^{1/3} + 0.0109 T_6^{2/3} + 0.00095 T_6\right] \ \mathrm{cm}^{-3}\ \mathrm{s}^{-1}.$$

Note that $3.1 \times 10^{-37} n_p^2 = 11.1 \times 10^{10} \rho^2 X_H^2$.

The lifetime of a proton in the Sun is

$$\frac{dn_p}{dt} = -\frac{n_p}{\tau_p\,(\mathrm{H})}; \qquad \tau_p\,(\mathrm{H}) \simeq 10^{10}\ \ \mathrm{yr}$$

PPI Chain

$$\mathrm{H}^1 + \mathrm{H}^1 \rightarrow \mathrm{D}^2 + e^+ + \nu_e \qquad r_{pp} = \lambda_{pp}\frac{H^2}{2}$$
$$\mathrm{D}^2 + \mathrm{H}^1 \rightarrow \mathrm{He}^3 + \gamma \qquad r_{pd} = \lambda_{pd} H D$$
$$\mathrm{He}^3 + \mathrm{He}^3 \rightarrow \mathrm{He}^4 + 2\mathrm{H}^1 \qquad r_{33} = \lambda_{33}\frac{\left(\mathrm{He}^3\right)^2}{2}$$

$$\frac{d\mathrm{D}}{dt} = \lambda_{pp}\frac{\mathrm{H}^2}{2} - \lambda_{pd}\mathrm{HD},$$
$$\frac{d\mathrm{He}^3}{dt} = \lambda_{pd}\mathrm{HD} - 2\lambda_{33}\frac{\left(\mathrm{He}^3\right)^2}{2},$$
$$\frac{d\mathrm{He}^4}{dt} = \lambda_{33}\frac{\left(\mathrm{He}^3\right)^2}{2}.$$

Step 1: deuterium equilibrium:

The deuterium equation is self-regulating in that D seeks an equilibrium value

$$\left(\frac{\mathrm{D}}{\mathrm{H}}\right)_{eq} = \frac{\lambda_{pp}}{2\lambda_{pd}} = \frac{\tau_p\,(D)}{2\tau_p\,(H)}.$$

Deuterium burning has $S(0) = 2.5\times10^{-4}$ keV-barn and $dS/dE = 7.9\times10^{-6}$ barn which is so fast that $\tau_p(\mathrm{D})$ is just a few seconds. At $T_6 \simeq 15$, $(\mathrm{D/H})_{eq} = 2.8\times10^{-18}$ and

$$(\mathrm{D/H}) = (\mathrm{D/H})_{eq} - \left[(\mathrm{D/H})_{eq} - (\mathrm{D/H})_0\right] e^{-t/\tau_p(\mathrm{D})}.$$

Step 2: He^3 equlibrium:

After deuterium equilibrium is reached, we have

$$\frac{d\mathrm{He}^3}{dt} = \lambda_{pp}\frac{\mathrm{H}^2}{2} - 2\lambda_{33}\frac{\left(\mathrm{He}^3\right)^2}{2},$$

which is another self-regulating equation leading to $(\mathrm{He}^3/\mathrm{H})_{eq} = \sqrt{\lambda_{pp}/2\lambda_{33}}$.

The cross section for r_{33} is not well known, about $S \approx 5000$ kev-barns.

Defining $x = \mathrm{He}^3/\mathrm{H}$ we have

$$\frac{dx}{dt} = \lambda_{33} H \left(x_{eq}^2 - x^2\right)$$

which, if H=constant is assumed, gives

$$x = x_{eq} \tanh\left(\frac{t}{\tau_3\left(3\right)_{eq}}\right).$$

At $T_6 = 15$ it takes about 10^6 yr for He^3 to build up to 99% of He$^3_{eq}$. One has $\tanh(2.647) \simeq 0.99$ and $\tau_3(3) \simeq 4 \times 10^5$ yr.

Energy production in PPI:

The $p-p$ reaction is quickly followed by the $d-p$ reaction so the net effect is 3H$\rightarrow$He$^3 + \nu_e$ at the rate r_{pp}. This liberates 6.936 MeV, minus an average of 0.263 MeV carried off by the neutrino. The energy liberation rate is then $1.069 \times 10^{-5} r_{pp}$.

The He^3-He^3 reaction liberates 12.858 MeV, so the total energy-generation rate is

$$\begin{aligned}\rho\epsilon_{ppI} =& 1.069 \times 10^{-5} r_{pp} + 2.060 \times 10^{-5} r_{33} \\ =& 2.099 \times 10^{-5} r_{pp},\end{aligned}$$

where the second equality is true after He^3 equilibrium since $r_{33} = r_{pp}/2$ then.

PPII and PPIII Chains

At temperatures in excess of $T_6 = 10$ (like the Sun's center) the reaction $\mathrm{He}^3{+}\mathrm{He}^4 \rightarrow \mathrm{Be}^7 + \gamma$ is important, even dominant, compared to $\mathrm{He}^3{+}\mathrm{He}^3 \rightarrow \mathrm{He}^4 + 2p$. In addition, Be7 can either capture an electron or a proton, leading to additional branching:

$$\begin{aligned} \mathrm{He}^3 + \mathrm{He}^4 &\rightarrow \mathrm{Be}^7 + \gamma \\ \mathrm{Be}^7 + e^- &\rightarrow \mathrm{Li}^7 + \nu_e \qquad\qquad \text{PPII} \\ \mathrm{Li}^7 + \mathrm{H}^1 &\rightarrow \mathrm{He}^4 + \mathrm{He}^4 \end{aligned}$$

or

$$\begin{aligned} \mathrm{Be}^7 + \mathrm{H}^1 &\rightarrow \mathrm{B}^8 + \gamma \\ \mathrm{B}^8 &\rightarrow \mathrm{Be}^8 + e^+ + \nu_e \qquad\qquad \text{PPIII} \\ \mathrm{Be}^8 &\rightarrow \mathrm{He}^4 + \mathrm{He}^4 \end{aligned}$$

These reactions lead to a set of differential equations

$$\begin{aligned} \frac{d\mathrm{H}}{dt} &= -\lambda_{pp}\mathrm{H}^2 - \lambda_{pd}\mathrm{HD} + \lambda_{33}\left(\mathrm{He}^3\right)^2 - \lambda_{17}\mathrm{HBe}^7 - \lambda'_{17}\mathrm{HLi}^7 \\ \frac{d\mathrm{D}}{dt} &= \lambda_{pp}\frac{\mathrm{H}^2}{2} - \lambda_{pd}\mathrm{HD} \\ \frac{d\mathrm{He}^3}{dt} &= \lambda_{pd}\mathrm{HD} - \lambda_{33}\left(\mathrm{He}^3\right)^2 - \lambda_{34}\mathrm{He}^3\mathrm{He}^4 \\ \frac{d\mathrm{He}^4}{dt} &= \lambda_{33}\frac{\left(\mathrm{He}^3\right)^2}{2} - \lambda_{34}\mathrm{He}^3\mathrm{He}^4 + 2\lambda_{17}\mathrm{HBe}^7 + \lambda'_{17}\mathrm{HLi}^7 \\ \frac{d\mathrm{Be}^7}{dt} &= \lambda_{34}\mathrm{He}^3\mathrm{He}^4 - \lambda_{e7}n_e\mathrm{Be}^7 - \lambda_{17}\mathrm{HBe}^7 \\ \frac{d\mathrm{Li}^7}{dt} &= \lambda_{e7}n_e\mathrm{Be}^7 - \lambda'_{17}\mathrm{HLi}^7. \end{aligned}$$

Step 1: deuterium equilibrium

After a time $\tau_p(\mathrm{D})$, a few seconds, deuterium comes into equilibrum and can be eliminated from the above:
$\lambda_{pd}\mathrm{HD} = \lambda_{pp}\mathrm{H}^2/2$.

Step 2: $\mathbf{Li^7}$ and Be7 equilibrium

This follows on timescales of years. In this case

$$\frac{d\left(\mathrm{Be}^7 + \mathrm{Li}^7\right)}{dt} = 0$$

and the equation for He^4 is simplified to

$$\frac{d\mathrm{He}^4}{dt} = \lambda_{33}\frac{\left(\mathrm{He}^3\right)^2}{2} + \lambda_{34}\mathrm{He}^3\mathrm{He}^4.$$

Note the last term has a positive sign. The hydrogen equation becomes

$$\frac{d\mathrm{H}}{dt} = -\frac{3}{2}\lambda_{pp}\mathrm{H}^2 + \lambda_{33}\left(\mathrm{He}^3\right)^2 - \lambda_{34}\mathrm{He}^3\mathrm{He}^4.$$

These can be solved under the assumptions that H and He^4 are constant, for times short to He^3 equilibration.

Step 3: $\mathbf{He^3}$ equilibrium

$$\mathrm{He}^3_{eq} = (2\lambda_{33})^{-1}\left[\sqrt{\left(\lambda_{34}\mathrm{He}^4\right)^2 + 2\lambda_{pp}\lambda_{33}\mathrm{H}^2} - \lambda_{34}\mathrm{He}^4\right].$$

Follows more or less what was found for PPI. The competition for He^3 determines which of PPI or PPII and PPIII dominates.

$$\frac{\mathrm{PPI}}{\mathrm{PPII} \;\; + \;\; \mathrm{PPIII}} = \frac{r_{33}}{r_{44}} = \frac{\lambda_{33}\mathrm{He}^3}{2\lambda_{34}\mathrm{He}^4}.$$

This branching ratio changes with stellar age, is 0 initially, reaches a maximum after He^3 reaches equilibrium, and decreases with increasing temperature.

After He^3 equilibrium,

$$\left(\frac{\mathrm{PPI}}{\mathrm{PPII}\ \ +\ \ \mathrm{PPIII}}\right)_{eq} = \frac{\sqrt{1+2\lambda_{pp}\lambda_{33}\left(\mathrm{H}/\mathrm{He}^4\right)^2/\lambda_{34}^2}-1}{4}$$

Define

$$\alpha = \frac{\lambda_{34}^2}{\lambda_{33}\lambda_{pp}}\left(\frac{\mathrm{He}^4}{\mathrm{H}}\right)^2 \simeq 1.2\times10^{17}\left(\frac{\mathrm{He}^4}{\mathrm{H}}\right)^2 e^{-100T_6^{-1/3}}.$$

Then

$$\left(\frac{\mathrm{PPI}}{\mathrm{PPII}\ \ +\ \ \mathrm{PPIII}}\right)_{eq} = \frac{\sqrt{1+2/\alpha}-1}{4}.$$

When $X=Y$, or He/H=1/4, the crossover from PPI to PPII and PPIII occurs when $\alpha = 1/12$, or when about $T_6 = 14$.

Once He^3 equilibrium is reached,

$$\begin{aligned}\frac{d\mathrm{He}^4}{dt} &= -\frac{1}{4}\frac{d\mathrm{H}}{dt} = \frac{\lambda_{pp}}{4}\mathrm{H}^2 + \frac{\lambda_{34}}{2}\mathrm{He}^3_{eq}\mathrm{He}^4 \\ &= \frac{\lambda_{pp}}{4}\mathrm{H}^2\left(1+\frac{2\lambda_{34}\mathrm{He}^3_{eq}\mathrm{He}^4}{\lambda_{pp}\mathrm{H}^2}\right) \\ &= \frac{\lambda_{pp}}{4}\mathrm{H}^2\left(1-\alpha+\alpha\sqrt{1+2/\alpha}\right).\end{aligned}$$

So when $\alpha\rightarrow0$, PPI operates by itself. When $\alpha\rightarrow\infty$, the quantity in parenthesis above is 2, reflecting that PPII and PPIII dominate and only 1 $p-p$ reaction is needed for a He^4 synthesis.

Energy production in PPI, PPII and PPIII:

Neutrino losses:

1. PPI: $2 \times 0.263/26.73 = 1.97\%$
2. PPII: $(0.263 + 0.80)/26.73 = 3.98\%$
3. PPIII: $(0.263 + 7.2)/26.73 = 27.9\%$

Rate of energy production:

$$\rho\epsilon = \frac{d\mathrm{He}^4}{dt} \left(4M_H - M_{He}\right) c^2 \left(0.9803 F_I + 0.9602 F_{II} + 0.721 F_{III}\right).$$

Obviously,

$$\frac{F_I}{1 - F_I} = \frac{\sqrt{1 + 2/\alpha} - 1}{4}$$

and

$$F_I = \frac{\sqrt{1 + 2/\alpha} - 1}{\sqrt{1 + 2/\alpha} + 3}.$$

Similarly, using

$$\tau_e\left(\mathrm{Be}^7\right)^{-1} = \lambda_{e7} n_e = 7.05 \times 10^{-33} n_e T_6^{-1/2}\ \mathrm{s}^{-1},$$

$$\tau_p\left(\mathrm{Be}^7\right)^{-1} = \lambda_{17} n_p = 6.3 \times 10^{-17} n_p T_6^{-2/3} e^{-102.65 T_6^{-1/3}}.$$

$$F_{II} = \left(1 - F_I\right) \frac{\tau_p\left(\mathrm{Be}^7\right)}{\tau_p\left(\mathrm{Be}^7\right) + \tau_e\left(\mathrm{Be}^7\right)}.$$

PPII takes over at about $T_6 = 14$, PPIII at about $T_6 = 23$.

The energy generation efficiency times the proton-proton rate ranges from 0.98 at low temperatures to 1.44 at high temperatures. However, depending on the composition, the factor at intermediate temperatures can be as large as 1.9.

Neutrino fluxes at earth from Be^7 are about 10^{10} ν cm^{-2} s^{-1} and from B^8 are about 2×10^7 ν cm^{-2} s^{-1}. Note that both D and Li^7 are completely destroyed in stellar burning.

CNO bi-cycle

Bethe and von Weisäcker in 1938 demonstrated that proton captures on C and N nuclei could compete with $p-p$ reactions at moderate temperatures. They also showed that the overall C + N abundance would not change as a result.

Later it became apparent that O can also participate, turning the cycle into a bi-cycle.

$$
\begin{aligned}
C^{12} + H^1 &\rightarrow N^{13} + \gamma \\
N^{13} &\rightarrow C^{13} + e^+ + \nu_e \qquad\qquad \tau = 870 \ \ \mathrm{s} \\
C^{13} + H^1 &\rightarrow N^{14} + \gamma \\
N^{14} + H^1 &\rightarrow O^{15} + \gamma \\
O^{15} &\rightarrow N^{15} + e^+ + \nu_e \qquad\qquad \tau = 178 \ \ \mathrm{s} \\
N^{15} + H^1 &\rightarrow C^{12} + He^4 \\
\text{or, branching ratio } 4 \times 10^{-4} & \\
N^{15} + H^1 &\rightarrow O^{16} + \gamma \\
O^{16} + H^1 &\rightarrow F^{17} + \gamma \\
F^{17} &\rightarrow O^{17} + e^+ + \nu_e \qquad\qquad \tau = 95 \ \ \mathrm{s} \\
O^{17} + H^1 &\rightarrow N^{14} + He^4
\end{aligned}
$$

Of these reactions, only the last is believed to be a resonant reaction. Near $T_6 = 25$, a characteristic CNO temperature, the sequence of increasing lifetimes of these reactions is τ_{15}, τ_{13}, τ_{12}, τ_{17}, τ_{14} and τ_{16}.

The relatively short beta-lifetimes means that N^{13}, O^{15} and F^{17} can be removed from consideration. The small value of the branching ratio means that first the CN cycle comes to equilibrium and then, possibly, the ON cycle. Consider first the CN cycle:

$$\begin{aligned}\frac{d\mathrm{C}^{12}}{dt} &= -\frac{\mathrm{C}^{12}}{\tau_{12}} + \frac{\mathrm{N}^{14}}{\tau_{14}}\\ \frac{d\mathrm{C}^{13}}{dt} &= \frac{\mathrm{C}^{12}}{\tau_{12}} - \frac{\mathrm{C}^{13}}{\tau_{13}}\\ \frac{d\mathrm{N}^{14}}{dt} &= \frac{\mathrm{C}^{13}}{\tau_{13}} - \frac{\mathrm{N}^{14}}{\tau_{14}}.\end{aligned}$$

This equation has the form $\dot{\vec{U}} = [\Lambda]\vec{U}$. If the nuclear lifetimes are constant, the solution consists in finding the three eigenvalues of $[\Lambda]$, say λ_1, λ_2 and λ_3. These satisfy

$$[\Lambda]\,\vec{U}_1 = \lambda_1\vec{U}_1, \qquad [\Lambda]\,\vec{U}_2 = \lambda_2\vec{U}_2, \qquad [\Lambda]\,\vec{U}_3 = \lambda_3\vec{U}_3.$$

$\vec{U}$ is a linear combination of the three eigenvectors:

$$\vec{U}\,(t) = Ae^{\lambda_1 t}\vec{U}_1 + Be^{\lambda_2 t}\vec{U}_2 + Ce^{\lambda_3 t}\vec{U}_3.$$

The constants A, B and C are determined by initial conditions: $\vec{U}(0) = A\vec{U}_1 + B\vec{U}_2 + C\vec{U}_3$.

The eigenvalues satisfy

$$det\left[[\Lambda] - \lambda\left[\vec{\mathbf{1}}\right]\right] = 0$$

which leads to the secular equation

$$\begin{vmatrix} -\left(\frac{1}{\tau_{12}} + \lambda\right) & 0 & \frac{1}{\tau_{14}} \\ \frac{1}{\tau 12} & -\left(\frac{1}{\tau_{13}} + \lambda\right) & 0 \\ 0 & \frac{1}{\tau_{13}} & -\left(\frac{1}{\tau_{14}} + \lambda\right) \end{vmatrix} = 0$$

The eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{\Delta - \Sigma}{2}, \qquad \lambda_3 = -\frac{\Delta + \Sigma}{2},$$

where

$$\Sigma = \frac{1}{\tau_{12}} + \frac{1}{\tau_{13}} + \frac{1}{\tau_{14}},$$

$$\Delta = \sqrt{\Sigma^2 - 4\left(\frac{1}{\tau_{12}\tau_{13}} + \frac{1}{\tau_{12}\tau_{14}} + \frac{1}{\tau_{13}\tau_{14}}\right)}.$$

The eigenvectors are

$$\vec{U}_1 = \frac{1}{\tau_{12} + \tau_{13} + \tau_{14}} \begin{pmatrix} \tau_{12} \\ \tau_{13} \\ \tau_{14} \end{pmatrix}$$

$$\vec{U}_2 = \begin{pmatrix} 1 \\ \frac{1/\tau_{12}}{1/\tau_{13} + (\Delta - \Sigma)/2} \\ -1 - \frac{1/\tau_{12}}{1/\tau_{13} + (\Delta - \Sigma)/2} \end{pmatrix}$$

$$\vec{U}_3 = \begin{pmatrix} 1 \\ -1 - \frac{1/\tau_{12} - (\Sigma + \Delta)/2}{1/\tau_{14}} \\ \frac{1/\tau_{12} - (\Sigma + \Delta)/2}{1/\tau_{14}} \end{pmatrix}$$

The first eigenvector represents the equilibrium abundance ratios in the CN cycle, i.e., $C^{12}_{eq} = \tau_{12}/(\tau_{12} + \tau_{13} + \tau_{14})$. The first eigenvalue is 0, meaning that if the abundances have these ratios, they don't change with time. The other two eigenvectors show how the abundances change with time:

$$\begin{pmatrix} C^{12}(t) \\ C^{13}(t) \\ N^{14}(t) \end{pmatrix} = \begin{pmatrix} C^{12}_{eq} \\ C^{13}_{eq} \\ N^{14}_{eq} \end{pmatrix} + B e^{\lambda_2 t} \vec{U}_2 + C e^{\lambda_3 t} \vec{U}_3.$$

The sum of the components of $\vec{U}_2$ and $\vec{U}_3$ vanish so that the abundance of CN nuclei don't change with time. One can show that

$$\begin{aligned} B + C =& \mathrm{C}^{12}(0) - \mathrm{C}^{12}_{eq}, \\ BU_{22} + CU_{32} =& \mathrm{C}^{13}(0) - \mathrm{C}^{13}_{eq}. \end{aligned}$$

The solution has the characteristic that $\lambda_2 \simeq 1/\tau_{12}$ and $\lambda_3 \simeq 1/\tau_{13}$. It is also clear that since N^{14} has the longest decay time the net effect of the CN cycle is to process C isotopes into N^{14}. The timescale for CN equlibrium is set by the slower of τ_{12} and τ_{13}, i.e., τ_{12}. This time is still much faster than the time of interchange of nuclei into the ON cycle.

The ON cycle can be solved by considering the leakage out of and into the CN cycle. The $\mathrm{N}^{15} + p$ reaction produces an O^{16}, adding it to the ON cycle, and the $\mathrm{O}^{17} + p$ reaction removes a nucleus from the ON cycle. Each reaction involves a fraction $\tau_{14}/(\tau_{14} + \tau_{15} + \tau_{16}) = 0.985 \simeq 1$. Therefore we can write the ON equations as

$$\frac{d}{dt}\begin{pmatrix} \mathrm{N}^{14} \\ \mathrm{O}^{16} \\ \mathrm{O}^{17} \end{pmatrix} = \begin{pmatrix} -\gamma/\tau_{14} & 0 & 1/\tau_{17} \\ \gamma/\tau_{14} & -1/\tau_{16} & 0 \\ 0 & 1/\tau 16 & -1/\tau_{17} \end{pmatrix} \begin{pmatrix} \mathrm{N}^{14} \\ \mathrm{O}^{16} \\ \mathrm{O}^{17} \end{pmatrix}.$$

This equation has the same form as the CN equation and has the same solution:

$$\begin{pmatrix} \mathrm{N}^{14}(t) \\ \mathrm{O}^{16}(t) \\ \mathrm{O}^{17}(t) \end{pmatrix} = \begin{pmatrix} \mathrm{N}^{14}_{eq} \\ \mathrm{O}^{16}_{eq} \\ \mathrm{O}^{17}_{eq} \end{pmatrix} + Be^{\lambda_2 t}\vec{U}_2 + Ce^{\lambda_3 t}\vec{U}_3.$$

The equilibrium concentration ratios are

$$\left(\frac{\mathrm{O}^{17}}{\mathrm{O}^{16}}\right)_{eq} = \frac{\tau_{17}}{\tau_{16}}, \qquad \left(\frac{\mathrm{O}^{16}}{\mathrm{N}^{14}}\right)_{eq} = \gamma\frac{\tau_{16}}{\tau_{14}}.$$

Thus all the CNO nuclei are converted to N^{14} if ON comes to equilibrium.

At relatively high temperatures, $T_6 > 25$, it is the case that $\tau_{17} < \tau_{12}$, implying that O^{17}/O^{16} come into equilibrium as fast as the CN equilibrium time. Furthermore, this ratio is $<< 1$, so all O is O^{16} and all CN are N^{14}. The interchange between the cycles is simplified:

$$\frac{d\mathrm{N}^{14}}{dt} \simeq -\gamma\frac{\mathrm{N}^{14}}{\tau_{14}} + \frac{\mathrm{O}^{16}}{\tau_{16}} \simeq -\frac{d\mathrm{O}^{16}}{dt},$$

with $N^{14}(t)+\mathrm{O}^{16}(t) \simeq N_{CN}+N_O$. The solution can be written as

$$\mathrm{N}^{14}(t) = \mathrm{N}^{14}_{eq} + \left(N_{CN}(0) - \mathrm{N}^{14}_{eq}\right) e^{-(\gamma/\tau_{14}+1/\tau_{16})t}.$$

Energy generation on the CNO cycle

For the equilibrated CN cycle,

$$\rho\epsilon_{CN,eq} = \left(5.53\frac{\mathrm{C}^{12}_{eq}}{\tau_{12}} + 12.1\frac{\mathrm{C}^{13}_{eq}}{\tau_{13}} + 22.5\frac{\mathrm{N}^{14}_{eq}}{\tau_{14}}\right) \times 10^{-6}\ \mathrm{erg\ cm^{-3}\ s^{-1}}$$

$$= 4.080\frac{N_{CN}}{\tau_{12}+\tau_{13}+\tau_{14}} \times 10^{-5}\ \mathrm{erg\ cm^{-3}\ s^{-1}}$$

$$= 8 \times 10^{27}\rho X_H X_{CN} T_6^{-2/3} e^{-152.31 T_6^{-1/3}}\ \mathrm{erg\ cm^{-3}\ s^{-1}}.$$

One can also evaluate the energy generation during the period the CN cycle is coming into equilibrium with the result that the initial rate of energy generation could be 30 times larger than the equilibrated value, although it rapidly decreases on timescales of thousands of years.

For the CNO cycle, we find

$$\rho\epsilon_{CNO} \simeq \left(40.8\frac{N_{CN}}{\tau_{14}} + 5.8\frac{N_{CNO}-N_{CN}}{\tau_{16}}\right)\times 10^{-6}\ \mathrm{erg\ cm^{-3}\ s^{-1}}.$$

Note that near $T_6 \simeq 25$ that the effective temperature exponent is $n \simeq 16.7$, so that a factor of 3 in energy generation is compensated by only a 7% change in the temperature.

In most stars, both the pp and CNO cycles operate simultaneously. Comparing the energy generation rates as a function of temperature indicates that for $X_{CN}/X_H \simeq 0.02$ the rates are equal for $T_6 \simeq 18$.

Triple alpha process

Since $A = 5$ and $A = 8$ have no stable nuclei, a problem exists in synthesizing elements beyond He. This is largely why these elements are not produced in the Big Bang. Hoyle suggested that the C^{12} compound nucleus formed by the fusion of an alpha particle and a Be^8 nucleus had an excitation energy near a resonance in C^{12}, although that resonance had not yet been detected. He showed that, although Be^8 has a very short lifetime, about 3×10^{-16} s ($\Gamma = 2.5$ eV), it could survive long enough for a collision with an alpha particle. The enhanced lifetime, compared to particle crossing times, is because Be^8 is unstable by only 92 keV. The heightened interaction due to the resonance is sufficient to allow the process to succeed.

$$
\begin{aligned}
N\left(\mathrm{Be}^8\right) \simeq & n_\alpha^2 \omega \frac{h^2}{(2\pi\mu kT)^{3/2}} e^{-E_r/kT} \\
\simeq & 2 \times 10^{-33} n_\alpha^2 T_8^{-3/2} \times 10^{-4.64/T_8} \mathrm{cm}^{-3}.
\end{aligned}
$$

The resonance in C^{12} is at an energy of 278 keV above the combined mass of Be^8 and He^4 and has a spin of 0 and a + parity. This state can't directly decay back to the ground state because it is also 0^+. It can decay to a lower state of 2^+, and then to the ground state.

One finds

$$1/\tau_{3\alpha} = 5.7 \times 10^{-7} \frac{\rho^2 X_H^2}{T_8^3} e^{-42.94/T_8} \ \ \mathrm{s}^{-‘}$$

The effective $n = 42.9/T_8 - 3$.