Homework # 4

The equation of state

$$P = K\rho^2 \left(1 + \frac{\rho}{\rho_0}\right)$$

gives the enthalpy h defined by $dh = dP/\rho$

$$h = 2K\rho \left(1 + \frac{3}{4}\frac{\rho}{\rho_0}\right)$$

if h vanishes where P and ρ vanish. This is a quadratic equation for $\rho(h)$.

Note that hydrostatic equilibrium with this equation of state has an analytic solution, which can be seen as follows: We know that the gradient of ρ vanishes near the origin, so the behavior must be quadratic there. Assume for the moment that

$$\rho = \beta\rho_0 \left(1 - \alpha r^2\right) + \dots$$

Then we have

$$m\left(r\right) = \int_0^r 4\pi\rho r^2 dr = \frac{4\pi}{3}\beta\rho_0 r^3 \left(1 - \frac{3}{5}\alpha r^2\right).$$

Hydrostatic equilibrum gives

$$\frac{1}{\rho}\frac{dP}{dr} = -\,4K\left(1 + \frac{3}{2}\beta\left(1 - \alpha r^2\right)\right)\beta\rho_0\alpha r$$

$$= -\frac{Gm}{r^2} = -\,\frac{4\pi G}{3}\beta\rho_0 r\left(1 - \frac{3}{5}\alpha r^2\right).$$

This is therefore a solution, since each size has a term linear in r and one cubic in r. Equating terms of the same dimension in r, one finds

$$\alpha = -\frac{2}{15}\frac{\pi G}{K}, \qquad \beta = 1.$$

Therefore ρ_0 is also the central density. We want the density to vanish for $r = R$ so $R^2 = \alpha^{-2}$. The total mass becomes

$$M = \frac{8\pi}{25}\rho_0 \left(\frac{9K}{2\pi G}\right)^{3/2}.$$

The values $K = 10^{14}$ cgs and $\rho_0 = 4$ g cm^{-3} give 1 $M_\odot$ and 1 $R_\odot$ for the star's mass and radius. (R determines α and therefore K, and then the equation for M determines ρ_0.)