Solutions to Final Exam

1. The basic relation to be used relates the luminosity to the available energy:

$$L = 4\pi R^2 \sigma T_s^4 = -\frac{dE_{th}}{dt}$$

where R is the radius, T_s is the surface temperature and E_{th} is the thermal energy

$$E_{th} \simeq \frac{3}{2}\frac{MN_o}{A}k_B T_i$$

where M is the mass and T_i is the average internal temperature. $A = 12$ is the atomic weight of the white dwarf material. Conduction by degenerate electrons is so effective that there is essentially no temperature gradient in the interior, so $T_{base} = T_i = 10T_s$. We thus have the equation

$$\frac{dT_s}{T_s^4} = -\frac{4\pi R^2 \sigma}{MN_o k_B}\frac{2}{3}\frac{1}{10}dt$$

Assuming the initial temperature is large compared to 3000 K, the left-hand side upon integration is $T_s^{-3}/3$. Therefore

$$t = \frac{5MN_o k_B}{4\pi R^2 A \sigma T_s^3} \simeq 1.3 Gyr$$

if we use $M = 2 \cdot 10^{33}$ g and $R = 5 \cdot 10^8$ cm.

2. The diffusion time from a uniformly dense sphere is

$$\tau_d = \frac{3R^2}{\pi^2 c \lambda_\nu}$$

where the mean free path $\lambda_n u = (\sigma_\nu n)^{-1}$ and the number density of absorbers is n. With $n = \rho N_o = 10^{15} \times 6 \cdot 10^{23}$ cm^{-3} and $R = 20$ km we find

$$\tau_d = 2.5 \cdot 10^{-4} E_\nu^2 \text{ s.}$$

The mean neutrino energy is $E_\nu = 3\mu_\nu/4$ where μ_ν is the neutrino chemical potential (the neutrinos are degenerate). We then have

$$E_\nu = \frac{3}{4}\hbar c \left(6\pi^2 n Y_\nu\right)^{1/3} \simeq 200 \text{ MeV}$$

which gives $\tau_d \simeq 10$ s.

The luminosity with the stated approximations will be

$$L_\nu = 6 \times 4\pi R^2 \sigma T_\nu^4.$$

In addition, we can estimate the total emitted energy as $3GM^2/5R \simeq 3{\cdot}10^{53}$ erg. If this is emitted over the diffusion time, then $L_\nu = 3\dot{1}0^{52}$ erg/s. From this the temperature can be estimated to be $3.6 \cdot 10^{10}$ K or about 3 MeV.

3. For coherent isotropic scattering,

$$S_\nu = \frac{\kappa_\nu B_\nu + \sigma_\nu J_\nu}{\kappa_\nu + \sigma_\nu}.$$

Integrating the transfer equation over μ gives

$$\frac{1}{2}\frac{dF_\nu}{d\tau_\nu} = 2\frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}\left(J_\nu - B_\nu\right).$$

In radiative equilibrium, no net energy is gained or lost at any optical depth, so both sides equal zero when integrated over frequency. Thus $F =$ constant.

If we can write

$$I\left(\mu, \tau\right) = I_0\left(\tau\right) + I_1\left(\tau\right)\mu + I_2\left(\tau\right)\mu^2 + \ldots,$$

then the first moment becomes

$$J\left(\tau\right) = \frac{1}{2}\int_{-1}^{1} I\left(\mu, \tau\right) d\mu = I_0\left(\tau\right) + \frac{1}{3}I_2\left(\tau\right) + \cdots.$$

Substituting I into the gray transfer equation

$$\mu\frac{dI\left(\mu, \tau\right)}{d\tau} = I\left(\mu, \tau\right) - J\left(\tau\right)$$

gives

$$\mu\frac{dI_0\left(\tau\right)}{d\tau} + \mu^2\frac{dI_1\left(\tau\right)}{d\tau} + \mu^3\frac{dI_2\left(\tau\right)}{d\tau} + \cdots = I_1\left(\tau\right)\mu + I_2\left(\tau\right)\left(\mu^2 - \frac{1}{3}\right) + \cdots.$$

Collecting the first 3 terms of equal powers in μ give, respectively, $I_2(\tau) = 0$, $dI_0(\tau)/d\tau = I_1(\tau)$, and $dI_1(\tau)/d\tau = I_2(\tau)$. Therefore I_1 is a constant and

$I_0(\tau) = C + \tau I_1$. Furthermore, collecting terms of higher powers in μ show that $I_i = 0$ for $i \geq 2$. Thus

$$I\left(\mu, \tau\right) = C + I_1\left(\tau + \mu\right).$$

The boundary condition is $I(-1, 0) = 0$ which implies that $C = I_1$. The first moment of I is the flux

$$F\left(\tau\right) = 2\int_{-1}^{1} \mu I\left(\mu, \tau\right) d\mu = \frac{4}{3} I_1\left(\tau\right) + \cdots.$$

I_1 is a constant, showing that the flux is constant, and $I_1 = (3/4)F$. The complete solution is

$$I\left(\mu, \tau\right) = \frac{3F}{4}\left(1 + \mu + \tau\right).$$

Limb darkening is the phenomenon that I varies with μ, namely that the ratio $I(0, 0)/I(1, 0) < 1$. In our case the ratio is 1/2.

The Eddington approximation is sometimes stated as $J(0) = F(0)/2$. We have $J(\tau) = (3F/4)(1 + \tau)$ and $J(0) = 3F/4$ so the Eddington approximation in this form is not realized. However, the other form is $K = 3J$ and

$$K\left(\tau\right) = \frac{1}{2}\int_{-1}^{1} I\left(\mu, \tau\right)\mu^2 d\mu = \frac{I_0\left(\tau\right)}{3} + \frac{I_2\left(\tau\right)}{5} + \cdots = \frac{I_0\left(\tau\right)}{3}.$$

Since $J(\tau) = I_0(\tau)$ this form of the Eddington approximation is realized.

4. The diffusion equation, or stellar structure luminosity equation, is

$$L = -4\pi R^2 \frac{4ac}{3\kappa\rho} T^3 \frac{dT}{dr},$$

where κ is the opacity. For electron scattering, $\kappa = \kappa_o$ is a constant. For radiation-dominated matter, $p = (a/3)T^4$ and $\epsilon\rho = aT^4$ where $\epsilon\rho$ is the energy density and ρ is the mass density of baryons. For a total mass, the total energy content is then $E = \epsilon M \propto aT^4 M/\rho$. By dimensional analysis, the diffusion equation becomes

$$L \propto \frac{R^2 c}{\kappa_o}\frac{aT^4}{\rho R} = \frac{Rc}{\kappa_o}\frac{E}{M}.$$

Supernovae have additional heat sources in the expanding ejecta due to radioactivity. The radioactive decay time for Ni→Co→ Fe is 7 d and 77 d, respectively.

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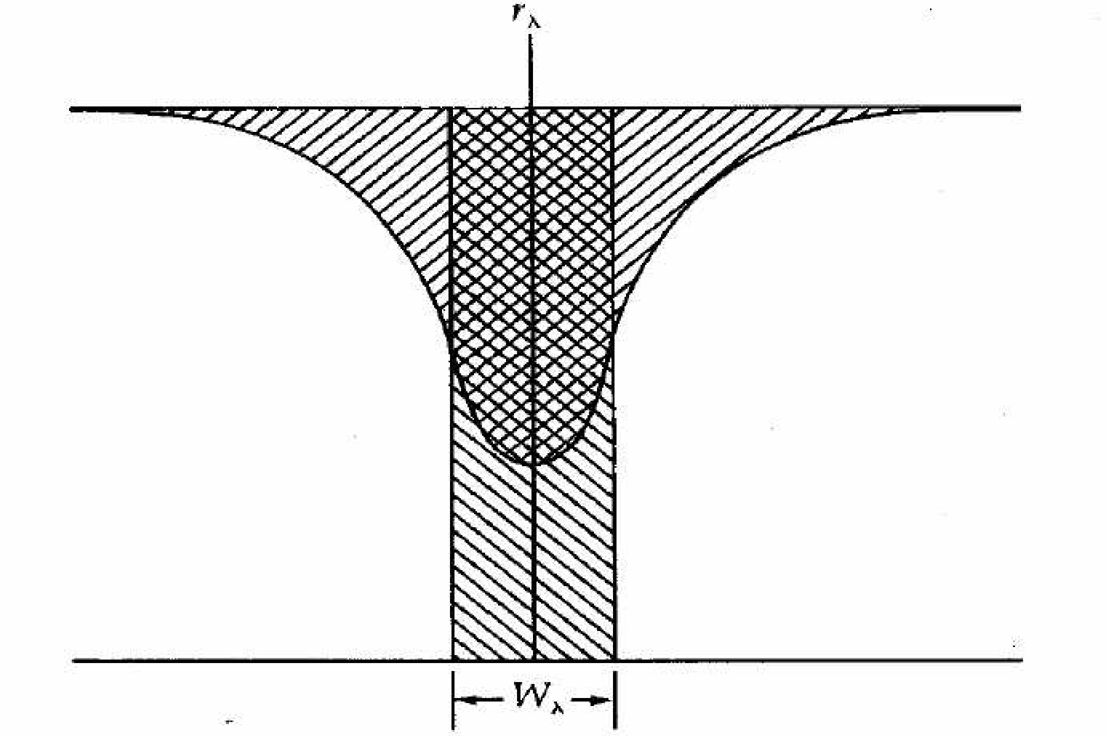

The gray atmosphere result using the Eddington approximation leads to an outgoing intensity ($\mu = +1$) of the form

$$I(\tau) = \frac{3F}{4}\left(\tau + \frac{2}{3}\right)$$

where τ refers to the optical depth. We apply this to a model in which a spectral line is produced by superimposing a cool gas between us and a continuum source. The line intensity incident on the base of the cool gas is the same as the emergent intensity in the neighboring continuum. Therefore

$$\frac{3F_C}{4}\left(0 + \frac{2}{3}\right) = \frac{3F_\nu}{4}\left(\tau + \frac{2}{3}\right).$$

The residual flux is defined as F_ν/F_C or

$$r_\nu = \frac{F_\nu}{F_C} = \left(1 + \frac{3\tau}{2}\right)^{-1}.$$

The equivalent width is

$$W = \int (1 - r_\nu)\, d\nu = \frac{3}{2}\int \frac{\tau}{1 + 3\tau/2} d\nu.$$

The optical depth is given by

$$\tau\left(\nu\right) = SNH$$

where S is the Einstein coefficient, N is the column density of absorbers, and H is the Voigt function. If the line is weak then $H \propto e^{-\left(\nu - nu_0\right)^2}$ where ν_0 is the line center frequency. If the number of absorbers is small the leading order term in the equivalent width integral is linear in N.

6 If we assume instantaneous recycling, and no infall, the mass of gas changes with time like

$$\frac{dm_g}{dt} = -\left(1 - R\right)\psi$$

where ψ is the star formation rate and R is the return fraction (fraction of a star returned to ISM when a star dies). For this problem, it would be OK to neglect R. If the "yield" p is the fraction of a star converted to metals and ejected when a star dies, the amount of metals in the gas changes as

$$\frac{d\left(Zm_g\right)}{dt} = \left(p - Z\right)\left(1 - R\right)\psi.$$

Thus

$$dZ = -pdm_g/m_g.$$

Integrating this gives

$$Z\left(t\right) = -p\ln\left(m_g\left(t\right)/m_g\left(0\right)\right),$$

where $Z(0) = 0$.

If we now suppose that $\psi = \nu m_g$, the first equation says that

$$dm_g/m_g = -\left(1 - R\right)\nu dt$$

which implies

$$dZ = -p\nu\left(1 - R\right)dt.$$

The closed box model predicts that the number of low-metallicity low-mass stars is much larger than observed. This problem can generally be resolved by allowing for continuous infall of metal-poor gas into the galactic disc.

7. For the Ba isotopes, clearly isotopes 130 and 132 can only be made by the p-process. Since their abundances are so small, the contribution of the p-process to other isotopes of Ba can be inferred to be negligible and can be ignored.

Ba isotopes 134 and 136 are shielded from the r-process by the r-process isotopes of Xe with the same mass. So 134 and 136 are s-only elements. Isotopes 135, 137 and 138 are produced by both s- and r-processes. None of the Ba isotopes are r-process only, unlike the isotopes 134 and 136 of Xe.

For slow neutron capture, the beta decays occur so quickly that neutron-capture equilibrium occurs between magic numbers. Thus the product $\sigma_i N_i$ is a constant for s-only elements. From the table, using the fraction f_i in place of the abundance N_i, one gets the product 0.079 for both 134 and 136. For the other Ba isotopes (135, 137, 138), this product is constant if the abundance refers to the s-process contribution. The amount produced by the r-process is then the total minus the amount produced by the s-process. Thus, the fraction of isotope i produced by the r-process is

$$\frac{N_{ir}}{N_i} = 1 - \frac{N_{is}}{N_i} = 1 - \frac{\sigma_i N_{is}}{\sigma_i N_i} = 1 - \frac{0.079}{\sigma_i N_i},$$

where we used the $\sigma_i N_{is}$ value of 0.079 as indicated by the s-only isotopes 134 and 136. For isotope 135 we find

$$\frac{N_{135r}}{N_{135}} = 1 - \frac{0.079}{0.066 \times 8.2} = 0.85.$$

For isotope 137 we find

$$\frac{N_{137r}}{N_{137}} = 1 - \frac{0.079}{0.112 \times 1.2} = 0.41.$$

For isotope 138 we find

$$\frac{N_{138r}}{N_{138}} = 1 - \frac{0.079}{0.717 \times 2.0} = 0.94.$$