Solutions # 1

1. *Consider a non-interacting fermion gas. Derive the following formula for the entropy per baryon s*

$$ns = -\frac{g}{h^3}\int \left[f\ln f + (1-f)\ln(1-f)\right] d^3p, \qquad (1)$$

where f is the probability that a given momentum state will be occupied:

$$f = \left[1 + \exp\left(\frac{E-\mu}{T}\right)\right]^{-1}. \qquad (2)$$

Here, the energy of a non-interacting particle, in terms of its rest mass m and momentum p, is

$$E^2 = m^2c^4 + p^2c^2, \qquad (3)$$

and μ is the chemical potential. Show your work.

There are 2 ways to do this. First, from the notes, for a fermion gas

$$W = \Pi\frac{(2n)!}{N_i!\,(2n-N_i)!} = \Pi\frac{1}{f_i!\,(1-f_i)!}$$

since $f_i = N_i/(2n)$. The entropy is $\ln W$. Using Stirling's formula gives

$$S = \ln W = -\sum_i \left[f_i\ln f_i + (1-f_i)\ln(1-f_i)\right].$$

Converting the sum into an integral gives our result.

The other way is to use the expression

$$Tns = \epsilon - n\mu - p = \frac{4\pi g}{h^3}\int f\left[E - \mu - \frac{p^2}{3E}\right] d^3p$$
$$= \frac{4\pi gT}{h^3}\int f\left[\ln\frac{1-f}{f} - \frac{p^3}{3E}\right] d^3p$$
$$= -\frac{4\pi gT}{h^3}\int \left[(1-f)\ln(1-f) + f\ln f - \ln(1-f) + \frac{p^3}{3E}\right] d^3p.$$

The first two terms are what we want. The last two terms cancel; to see this integrate the 3rd term by parts using $u = \ln(1-f)$ and $dv = d^3p/3$, and $df/dp = -f(1-f)p/(ET)$.

2. *Determine the entropy per baryon of a room-temperature gas. Alternatively, you may compute the entropy density of a room-temperature gas. Show your work.*

A room temperature gas is composed of non-degenerate molecules of nitrogen and oxygen. Each gas contributes an entropy per baryon $5/2 - \psi_i$ in units of k_B where

$$n_i = g\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} e^{\psi_i}$$

is the number density. We'll use T=300 K, $g = 1$ and 1 atmosphere pressure= 10^6 erg cm^{-3}. For simplicity treat each molecule as having a molecular weight of 30. The number density is then

$$n = \frac{10^6}{300 \cdot 1.38e^{-16}} = 2.4 \cdot 10^{19} \ \text{cm}^{-3}.$$

The degneracy parameter is

$$\psi = \ln \left[\frac{n}{g} \left(\frac{2\pi\hbar^2}{mk_BT} \right)^{3/2} \right] = -15.7.$$

The entropy per baryon is about $18.2k_B$.

3. *Determine the entropy per baryon or the entropy density at the center of the Sun. Show your work.* For the Sun, assume 100% hydrogen for simplicity, $\rho = 75$ g cm^{-3}, $g = 1$ and $T = 15 \cdot 10^6$ K. Then $n = \rho N_0 = 4.5 \cdot 10^{25}$ cm^{-3}. The degeneracy parameter is

$$\psi = \ln \left[\frac{n}{g} \left(\frac{2\pi\hbar^2}{mT} \right)^{3/2} \right] = -12.4.$$

The entropy per baryon is about $15k_B$.

We also have to include the entropy of the electrons. The mass is 1837 times smaller but the number density is the same. The degeneracy parameter for the electrons is -1.1 and they contribute an entropy per baryon of $3.5k_B$ for a total of about $20k_B$.