Scaling of Stars

The stellar variables include L, M, R and T, where T is the core temperature. The structure equations imply the proportionalities:

$$\begin{aligned}
L \propto& \frac{RT^4}{\kappa\rho,} \\
L \propto& M\dot{\epsilon}, \\
T \propto& \mu\beta\frac{M}{R}, \\
\rho \propto& \frac{M}{R^3}, \\
\mu\beta \propto& \frac{(1-\beta)^{1/4}}{M^{1/2}}.
\end{aligned}$$

Note that

$$\frac{\partial \ln M}{\partial \ln \beta} = -\frac{2-\beta}{1-\beta}.$$

For high-mass stars

$$\kappa \propto (1+X)\,, \qquad \dot{\epsilon} \propto XZ\rho T^{20}.$$

For low-mass stars

$$\kappa \propto Z\,(1+X)\,\rho T^{-3.5}, \qquad \dot{\epsilon} \propto X^2\rho T^4.$$

For high-mass stars, use CNO cycle and electron scattering opacity. Also, $1-\beta$ does not vary rapidly with mass, so eliminate $\mu\beta$ as well as ρ and T:

$$L \propto M \frac{1-\beta}{1+X},$$
$$L \propto XZ \frac{M^{12}}{R^{23}} (1-\beta)^5 .$$

Then

$$R \propto [XZ(1+X)]^{1/23} (1-\beta)^{4/23} M^{11/23},$$
$$T_{eff} \propto \frac{L^{1/4}}{R^{1/2}} \propto \frac{(1-\beta)^{15/92}}{(XZ)^{1/46}(1+X)^{25/92}} M^{1/92},$$
$$T_{eff} \propto \frac{(1-\beta)^{7/46}}{(XZ)^{1/46}(1+X)^{6/23}} L^{1/92}.$$

Since $\partial \ln(1-\beta)/\partial \ln M \sim 2-4$ for $\beta \sim 0.8-1.0$, the dependence on $1-\beta$ should be included when looking at mass dependences.

For low-mass stars, use pp cycle and Kramer's opacity. Also, $1-\beta$ does vary rapidly with mass, so eliminate $1-\beta$, ρ and T:

$$R \propto \left(X^2 (1+X) Z\right)^{2/19} (\mu\beta)^{-7/19} M^{1/19},$$

$$L \propto \left(\frac{X^9}{(1+X)^{10} Z^{10}}\right)^{2/19} (\mu\beta)^{146/19} M^{104/19},$$

$$T_{eff} \propto X^{5/38} (Z(1+X))^{-6/19} (\mu\beta)^{40/19} M^{51/38},$$

$$T_{eff} \propto X^{-709/1976} (Z(1+X))^{-3/52} (\mu\beta)^{23/104} L^{51/208}.$$

1) As H consumed, X decreases and μ increases

$$\mu \simeq 4/\left(5X+3\right)$$

and β is nearly constant. So L increases and T_{eff} increases; stars evolve *up* the main sequence. This explains why in globular clusters the M-S turnoff luminosity $>> L_\odot$ even though $M \leq \mathrm{M}_\odot$. Also, the early Sun was less luminous, and cooler, than present. If initial (present) $X = 0.75(0.7)$,

$$\frac{L_{today}}{L_{initial}} \simeq 1.4, \quad \frac{T_{eff,today}}{T_{eff,initial}} \simeq 1.11,$$

$$\frac{T_{c,today}}{T_{c,initial}} \simeq 1.09, \quad \frac{R_{today}}{R_{initial}} \simeq 0.96.$$

2) Low-mass stars on the p-p cycle ($\nu = 4$) have R nearly independent of M: $R \propto M^{1/19}$ for Kramer's opacity. For high-mass stars on the CNO cycle, however, $R \propto M^{11/23}$ for electron scattering opacity.

3) Population II stars are characterized by low metal compositions, $Z < 0.001$. The pp cycle dominates energy production even for massive stars. For low-mass stars, $L \propto Z^{-20/19}$ for a given mass: the Population II M-S is shifted to higher L than the Population I M-S. Also, for a given M for low-mass stars, $T_{eff} \propto Z^{-6/19}$ implies a shift of the M-S to higher T_{eff}. But notice that for a given T_{eff}, L increases as $Z^{4/17}$ so the Population II M-S is shifted *below* the Population I M-S.

4) For a given M, L is larger for Population II than for Population I stars. Stellar lifetimes $\tau \propto M/L$ are nearly $\propto Z$ since $\kappa \propto Z(1+X)$. Thus, lifetimes of Population II stars are substantially less than Population I for a given mass. This is observed in H-R diagrams of globular clusters.