Nuclear Reactions and Resonances

Consider the reaction

$$a + X \rightarrow Z^* \rightarrow Y + b \qquad \text{or} \qquad (X\,(a,b)\,Y)$$

with a the projectile, X the target and b and Y the products. Z^* is an intermediary stage, the compound nucleus. Often there are many b, Y possibilities, and each has a probability of occuring, $\mathcal{P}_i = \tau/\tau_i$ where τ_i is the mean-life for reaction i and $\tau = [\sum \tau_i^{-1}]^{-1}$. Among the exit channels is the one in which b, Y=a, X which represents scattering.

Energy width Γ_i defined through the uncertainty principle,

$$\Gamma_i \tau_i = \hbar, \qquad \Gamma = \sum \Gamma_i.$$

The cross section can be written as

$$\sigma_{ab}\,(E) = \pi \left(\frac{\lambda}{2\pi}\right)^2 g \frac{\Gamma_a \Gamma_b}{\Gamma^2} f\,(E)$$

where

$$\pi \left(\frac{\lambda}{2\pi}\right)^2 = \frac{\pi\hbar^2}{2Em_b} = \frac{0.657\ \text{MeV}}{E}\frac{1}{\mu}\ \text{barns}$$

with 1 barn=10^{-24} cm^2. g is a spin-dependent factor. The so-called "maximum cross section" is geometrical. A particle a with linear momentum p approaching X with impact parameter b has a quantized angular momentum $bp = \ell\hbar$. Then the fractional cross section of the ring defined by ℓ and $\ell + 1$ is $\pi(\lambda/2\pi)^2(2\ell + 1)$ where $(\lambda/2\pi) = \hbar/p$.

The shape factor $f(E)$ is controlled by whether the reaction is resonant or non-resonant. For non-resonant reactions,

$$\sigma_{ab}\,(E) = \frac{S\,(E)}{E} e^{-b/\sqrt{E}}.$$

For resonant reactions, we have the Breit-Wigner formula

$$f(E) = \frac{\Gamma^2}{(E - E_r)^2 + (\Gamma/2)^2},$$

which is sharply peaked near the resonance energy E_r. Supposing that Γ doesn't vary much within the energy $\Gamma/2$ of E, and that f is sharply peaked, we can write the Maxwellian average of the cross section and velocity as

$$\begin{aligned} < \sigma_{ab} v > &= \frac{\pi \hbar^2 g}{2m} \sqrt{\frac{8}{\mu m_b}} (kT)^{-3/2} e^{-E_r/kT} \int_0^\infty \frac{\Gamma_a \Gamma_b dE}{(E - E_r)^2 + (\Gamma/2)^2} \\ &= \hbar^2 \left(\frac{2\pi}{m_b kT} \right)^{3/2} g \frac{\Gamma_a \Gamma_b}{\Gamma} e^{-E_r/kT} \\ &= 2.56 \times 10^{-13} \frac{(\omega\gamma)_r}{(\mu T_9)^{3/2}} e^{-11.605 E_r/T_9} \ \mathrm{cm}^3 \ \mathrm{s}^{-1}. \end{aligned}$$

Here we've used

$$\int_0^\infty \sigma_{ab}(E) \, dE = \frac{\hbar^2 \pi^2}{m E_r} \left[g \frac{\Gamma_a \Gamma_b}{\Gamma} \right] = \frac{\hbar^2 \pi^2}{m E_r} (\omega\gamma)_r \, .$$

This is useful since a poor resolution experiment only yields an integrated cross section. E_r and $(\omega\gamma)_r$ are in MeV. Note that near a temperature T_9 the effective temperature exponent for a resonant rate is

$$n = \frac{11.605 E_r}{T_9} - \frac{3}{2}.$$

Weak Interaction Rates

Fermi's Golden Rule #2:

$$\Gamma = \frac{2\pi}{\hbar}|H_{mi}|^2\rho.$$

Here H_{mi} is the perturbed part of the time-dependent Hamiltonian and ρ is the density of final states. The Fermi theory of weak interactions for the reaction $n \to p + e + \bar{\nu}$ has

$$H_{mi} = G_F \int \psi_p^* \psi_e^* \psi_{\bar{\nu}}^* \psi_n d^3 r.$$

The wave functions of the electron and antineutrino are plane waves with de Broglie wavlengths much greater than the nucleon dimension, so $\psi_e \psi_{\bar{\nu}} \approx 1$. We have

$$H_{pn} = G_F \int \psi_p^* \psi_n d^3 r.$$

The overlap integral is nearly unity for nucleon decay, but can be much less than one for other weak interactions, and zero for one in which β decay is not allowed. Here $G_F \simeq 1.4 \times 10^{-49}$ erg cm^3. The density of final states results in the transition rate

$$d\Gamma = \frac{2\pi}{\hbar} M^2 \rho_e \rho_{\bar{\nu}} \delta\left(E_e + E_{\bar{\nu}} - E_n + E_p\right) dE_e dE_{\bar{\nu}},$$

where $M^2 = \sum_{spin} |H_{pn}|^2/2$. Integrating over $E_{\bar{\nu}}$,

$$d\Gamma = \frac{2\pi}{\hbar} M^2 \rho_e \rho_{\bar{\nu}} dE_e.$$

Neglecting spin, the density of electron and antineutrino states:

$$\rho_e = \frac{4\pi p_e^2}{h^3}\frac{dp_e}{dE_e} = \frac{4\pi p_e E_e}{c^2 h^3}, \qquad \rho_{\bar{\nu}} = \frac{4\pi\left(E_n - E_p - E_e\right)^2}{h^3 c^3}.$$

Thus

$$d\Gamma = \frac{64\pi^4 G_F^2 M^2}{h^7} m_e^5 c^4 \sqrt{\epsilon^2 - 1}\epsilon \left(\epsilon_0 - \epsilon\right)^2 d\epsilon,$$

where

$$\epsilon = \frac{E_e}{m_e c^2}, \qquad \epsilon_0 = \frac{E_n - E_p}{m_e c^2}.$$

For neutron decay, $E_n - E_p \approx (m_n - m_p)c^2 = 1.297$ MeV and $\epsilon_0 \approx 2.53$. The integration over energy yields

$$\begin{aligned} f\left(\epsilon_0\right) =& \int_1^{\epsilon_0} \sqrt{\epsilon^2 - 1}\epsilon \left(\epsilon_0 - \epsilon\right)^2 d\epsilon \\ =& \sqrt{\epsilon_0^2 - 1} \left(\frac{\epsilon_0^4}{30} - \frac{3\epsilon_0^2}{20} - \frac{2}{15}\right) + \frac{\epsilon_0}{4} \ln\left[\epsilon_0 + \sqrt{\epsilon_0^2 - 1}\right], \end{aligned}$$

which is 1.64 for neutron decay. The cross section is $\sigma = \Gamma/v$.

For the proton-proton reaction, $\epsilon_0 = (2m_p - m_D)/m_e \simeq 2.33$. The cross section also involves the Coulomb barrier penetration, and M^2 is not unity:

$$\begin{aligned} \sigma_{pp} =& P\left(E\right)\sigma = \frac{2\pi e^2 \sigma}{\hbar v} e^{-2\pi e^2/\hbar v} \\ \sigma \simeq& 4 \times 10^{-49} \text{ cm}^2. \end{aligned}$$