Thermonuclear Reactions

Non-resonant Reaction Rates

Schrödinger equation:

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi + V(r)\,\Psi = E\Psi.$$

$\mu = m_1 m_2/(m_1+m_2)$, E is incident kinetic energy.

$$V(r) = \begin{array}{cc} Z_1 Z_2 e^2/r & r > R \\ -V_0 & r < R \end{array}$$

$$\Psi = \frac{\chi_\ell(r)}{r} Y_\ell^m(\theta,\phi)$$

Assume radial symmetry, ignore angular parts. Define χ:

$$-\frac{\hbar^2}{2\mu}\chi_\ell'' + \left[\frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V(r) - E\right]\chi_\ell(r) = -\frac{\hbar^2}{2\mu}\chi_\ell'' + f(r)\,\chi_\ell(r) = 0.$$

$$f(r) \quad \begin{array}{cc} < 0 & r > R_0 \\ > 0 & r < R_0 \end{array},$$

The turning point R_0 is defined by:

$$E = \frac{Z_1 Z_2 e^2}{R_0} + \frac{\ell(\ell+1)\hbar^2}{2\mu R_0^2}.$$

Define ϕ:

$$\chi_\ell(r) = A\mathrm{e}^{i\phi(r)/\hbar}; \qquad i\hbar\phi'' - \left(\phi'\right)^2 + 2\mu\left[E - V(r)\right] = 0.$$

Lowest order approximation: Neglect ϕ'':

$$\phi(r) \simeq \pm\sqrt{2\mu}\int_R^r \sqrt{E - V(r')}dr',$$

which is valid as long as r is not near R_0. The integrand is real for $r > R_0$ and imaginary for $r < R_0$. The constant A is set by normalization, so $\chi^*(r)\chi(r)$ gives the probability per unit radial distance that incoming nucleus is at r. The penetration factor $P_\ell(E,r)$ is then

$$P_\ell\left(E,r\right)=\frac{\chi^*\left(R\right)\chi\left(R\right)}{\chi^*\left(r\right)\chi\left(r\right)}\propto\exp\left(-2\int_R^r\sqrt{\frac{2\mu}{\hbar^2}\left[V\left(r'\right)-E\right]}dr'\right).$$

Setting $R\approx 0$, $r\approx R_0$ (the solution oscillates for $r>R_0$), and for the case $\ell=0$, the integral is

$$\int_0^{R_0}\sqrt{\frac{2\mu}{\hbar^2}\left[V\left(r'\right)-E\right]}dr'=\frac{\sqrt{2\mu E}}{\hbar}\frac{Z_1Z_2e^2}{E}\frac{\pi}{2}=\pi\frac{Z_1Z_2e^2}{\hbar v},$$

using $v^2=2E/\mu$. This forms the *Gamow factor.*

Cross Section: (Reactions/s)/(Incident Flux), units of area.

Number of reactions per unit volume per unit time is $n_1n_2v\sigma$ at a given energy (relative velocity v). Integrating over all energies and correcting for double counting of identical particles:

$$r=\int\int n_1\left(v_1\right)n_2\left(v_1\right)v\sigma\left(v\right)d^3v_1d^3v_2/\left(1+\delta_{1,2}\right).$$

Maxwellian distributions:

$$n_1\left(v_1\right)=n_1\left(\frac{m_1}{2\pi kT}\right)^{3/2}\mathrm{e}^{-\frac{m_1v_1^2}{2kT}}.$$

Note that

$$\int_0^\infty n_1\left(v_1\right)d^3v_1\equiv n_1.$$

The relative velocity v and the center-of-mass velocity V are:

$$v_1=V-\frac{m_2}{m_1+m_2}v,\qquad v_2=V+\frac{m_1}{m_1+m_2}v.$$

$$n_1\left(v_1\right)d^3v_1\ \ n_2\left(v_2\right)d^3v_2=$$

$$n_1n_2\left[\left(\frac{m_1+m_2}{2\pi kT}\right)^{3/2}\mathrm{e}^{-\frac{(m_1+m_2)V^2}{2kT}}d^3V\right]\left[\left(\frac{\mu}{2\pi kT}\right)^{3/2}\mathrm{e}^{-\frac{\mu v^2}{2kT}}d^3v\right].$$

Integration over d^3V of the first bracket is unity, and so

$$\begin{aligned} r =& \frac{n_1 n_2}{1+\delta_{1,2}} \int_0^\infty v\sigma\left(v\right)\left(\frac{\mu}{2\pi kT}\right)^{3/2} \mathrm{e}^{-\frac{\mu v^2}{2kT}} d^3v \\ =& \frac{n_1 n_2}{1+\delta_{1,2}} \sqrt{\frac{8}{\mu\pi kT}} \int_0^\infty \frac{E\sigma\left(E\right)}{kT} \mathrm{e}^{-\frac{E}{kT}} dE. \end{aligned}$$

The Gamow factor is most significant part of nuclear cross sections. Also, since maximum quantum mechanical geometrical cross section $\propto \lambda^2 \propto 1/E$, it is convenient to write

$$\sigma\left(E\right) = \frac{S\left(E\right)}{E} \mathrm{e}^{-b/\sqrt{E}},$$

where $S(E)$ is slowly varying. For $\ell = 0$, $A = \mu/m_b$,

$$b = \frac{\pi\sqrt{2\mu}Z_1 Z_2}{\hbar} = 31.3 Z_1 Z_2 \sqrt{A}\ \text{keV}^{1/2}.$$

$$\begin{aligned} r =& \frac{n_1 n_2}{1+\delta_{1,2}} \sqrt{\frac{8}{\mu\pi kT}} \frac{1}{kT} \int_0^\infty S\left(E\right) \mathrm{e}^{-E/kT - b/\sqrt{E}} dE \\ =& \frac{n_1 n_2}{1+\delta_{1,2}} \sqrt{\frac{8}{\mu\pi kT}} \frac{1}{kT} \mathrm{e}^{-3E_0/kT} S\left(E_0\right) \int_0^\infty \exp\left(-\left[\frac{E-E_0}{\Delta/2}\right]^2\right) dE \\ =& \frac{n_1 n_2}{1+\delta_{1,2}} \sqrt{\frac{2}{\mu\pi kT}} \frac{\Delta}{kT} \mathrm{e}^{-3E_0/kT} S\left(E_0\right). \end{aligned}$$

We approximated the integrand as a Gaussian with centroid

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} = 1.22 \left(Z_1^2 Z_2^2 A T_6^2\right)^{1/3}\ \text{keV}$$

and width (note $T_6 = T/10^6$ K)

$$\Delta = 4\sqrt{\frac{E_0 kT}{3}} = 0.75 \left(Z_1^2 Z_2^2 A T_6^5\right)^{1/6}\ \text{keV}.$$

Define

$$\tau = \frac{3E_0}{kT} = 42.5\left(\frac{Z_1^2 Z_2^2 A}{T_6}\right)^{1/3}$$

one has $\Delta = 4\sqrt{\tau}kT/3$ and

$$\begin{aligned} r =& \frac{n_1 n_2}{1+\delta_{1,2}} \frac{2}{3\mu} \frac{\tau^2}{9b} S\left(E_0\right) \mathrm{e}^{-\tau} \\ =& 4.5 \cdot 10^{14} \frac{n_1 n_2}{1+\delta_{1,2}} \frac{S\left(E_0\right)}{A Z_1 Z_2} \tau^2 \mathrm{e}^{-\tau}. \end{aligned}$$

Electron Screening:

Modified Coulomb potential because of electron screening:

$$\frac{Z_1 e}{r}\mathrm{e}^{-r/\lambda_D} \approx \frac{Z_1 e}{r} - \frac{Z_1 e}{\lambda_D}$$

with λ_D the Debye length

$$\lambda_D = \sqrt{\frac{kT}{4\pi\left(Z_1+1\right)e^2 n_e}} \simeq 9.2 \cdot 10^{-9} \sqrt{\frac{T_6}{\rho Y_e\left(Z_1+1\right)}} \text{ cm.}$$

Then reaction rate increased by

$$\exp\left(\frac{Z_1 Z_2 e^2}{kT\lambda_D}\right) \simeq \exp\left(0.17 Z_1 Z_2 \sqrt{\frac{\rho Y_e\left(Z_1+1\right)}{T_6^3}}\right).$$

Effective Thermonuclear Rate:

Near a temperature T_0, one can write $r = r_0 (T/T_0)^n$, where

$$n = \left.\frac{d\ln r}{d\ln T}\right|_{T_0} = \left.\frac{d\ln\tau}{d\ln T}\frac{d\ln r}{d\ln\tau}\right|_{T_0} = \left.\frac{\tau-2}{3}\right|_{T_0}.$$