Radiative Transfer

Specific Intensity I defined such that $I(r,\theta)d\Omega$ is the radiative energy flux (erg cm^{-2}s^{-1}) passing through $d\Omega$ (in steradians) around a colatitude angle θ (r, θ, ϕ space) at distance r. The angle θ is between the radial direction and the vertical distance z.

Energy transfer is assumed to be time-independent.

Corresponding to this is the energy density u:

$$u\left(r,\theta\right)d\Omega = \frac{I\left(r,\theta\right)}{c}d\Omega$$

Total energy density is

$$U\left(r\right) = \int u\left(r,\theta\right)d\Omega = \frac{2\pi}{c}\int_{-1}^{1} I\left(r,\mu\right)d\mu$$

where $\mu = \cos\theta$ and we assume azimuthal symmetry.

The total flux in the z-direction is $\mathcal{F}$ (erg cm^{-2}s^{-1}) is

$$\mathcal{F}\left(r\right) = \int I\left(r,\theta\right)\cos\theta d\Omega = 2\pi\int_{-1}^{1} I\left(r,\mu\right)\mu d\mu.$$

If I is constant, then $\mathcal{F} = 0$. Hence I varies with μ in order for energy to be transported.

Define mass emission coefficient $j(r,\theta)$ and mass absorption coefficient (opacity) $\kappa(r,\theta)$. Net energy removed over a distance s is

$$dI = j\left(r,\theta\right)\rho\left(r\right)ds - \kappa\left(r,\theta\right)\rho\left(r\right)I\left(r,\theta\right)ds.$$

If $j = 0$ and $\kappa\rho$ is constant, solution is

$$I \propto e^{-\kappa\rho s}$$

or simple attenuation. $(\kappa\rho)^{-1}$ is e-folding length for attenuation.

Radiative transfer equation is

$$\frac{1}{\rho}\frac{dI\left(r,\theta\right)}{ds} = j\left(r,\theta\right) - \kappa\left(r,\theta\right) I\left(r,\theta\right)$$

which is strictly true only in planar geometry.

If LTE, isotropy and spatial uniformity existed, then $dI/ds = 0$ and $I = j/\kappa$. Then the energy density is

$$U = \frac{2\pi}{c}\int_{-1}^{1} I d\mu = \frac{4\pi}{c} I = aT^4.$$

Thus

$$I = \frac{caT^4}{4\pi} = B\left(T\right), \qquad B_\nu\left(T\right) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1}.$$

Radiative transfer involves small departures from LTE.

Define source function S and optical depth τ:

$$S_\nu\left(r,\theta\right) = j_\nu\left(r,\theta\right)/\kappa\left(r,\theta\right); \qquad d\tau_\nu = -\kappa_\nu \rho dz$$

$$\mu\frac{dI_\nu\left(\mu,\tau_\nu\right)}{d\tau_\nu} = I_\nu\left(\mu,\tau_\nu\right) - S_\nu\left(\mu,\tau_\nu\right).$$

Multiply this through by $\exp(-\tau/\mu)$ to find

$$\frac{d}{d\tau}\left(e^{-\tau/\mu} I\right) = -e^{-\tau/\mu} S/\mu.$$

Integrate this from τ_0 to τ:

$$I(\tau,\mu) = e^{-(\tau_0-\tau)/\mu} I(\tau_0,\mu) + \int_\tau^{\tau_0} e^{-(t-\tau)/\mu} S(t)\, dt/\mu.$$

Look first at forward-directed radiation ($\mu \geq 0$), choose $\tau_0 \to \infty$:

$$I(\tau,\mu \geq 0) = \int_\tau^{\infty} e^{-(t-\tau)/\mu} S(t)\, dt/\mu.$$

For inward-directed radiation ($\mu < 0$), choose $\tau_0 = 0$:

$$I(\tau,\mu < 0) = \int_\tau^{0} e^{-(t-\tau)/\mu} S(t)\, dt/\mu.$$

Note that $I(0,\mu < 0) = 0$ if there is no incident radiation.

In deep interior, reasonable to expand S

$$S(t) = B(\tau) + (t-\tau)\left(\partial B/\partial\tau\right)_\tau,$$

where $B(\tau) = B[T(\tau)]$. With this, the previous two expressions become

$$I(\tau,\mu \geq 0) = B(\tau) + \mu\left(\partial B/\partial\tau\right)_\tau,$$

$$\begin{aligned} I(\tau,\mu < 0) =& B(\tau)\left[1 - e^{\tau/\mu}\right] \\ &+ \mu\left(\partial B/\partial\tau\right)_\tau \left[e^{\tau/\mu}(\tau/\mu - 1) + 1\right]. \end{aligned}$$

But $\mu < 0$ in the last expression, so $e^{\tau/\mu} \rightarrow 0$ for large τ, so that the first expression is then valid for **all** μ. Then the total flux is

$$\begin{aligned}\mathcal{F}(\tau) =& 2\pi \int_{-1}^{1} I(\tau, \mu)\, \mu d\mu \\ =& 2\pi \int_{-1}^{1} \left[B(\tau) + \mu \frac{\partial B(\tau)}{\partial \tau} \right] \mu d\mu = \frac{4\pi}{3} \frac{\partial B(\tau)}{\partial \tau}.\end{aligned}$$

$$\mathcal{F}_\nu = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{dT}{dr} \frac{\partial B_\nu}{\partial T}.$$

Integrating over frequency, this is equivalent to

$$\begin{aligned}\mathcal{F}(r) =& -\frac{4\pi}{3} \frac{1}{\kappa_R \rho} \frac{dT}{dr} \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu \\ =& -\frac{4ac}{3} \frac{1}{\kappa_R \rho} T^3 \frac{dT}{dr}.\end{aligned}$$

This equation thereby defines the Rosseland mean opacity κ_R, which is seen to be a suitably frequency-averaged opacity. This equation is known as Fick's Law of Diffusion, and is valid in regions with large optical depth. In our study of atmospheres, we must generalize this result to include small optical depth regions.

Luminosity defined to be $L(r) = 4\pi r^2 \mathcal{F}(r)$.