Nuclear Astrophysics

Nuclear energies: The Liquid Drop Model.

Nuclei have internal baryon densities $n \simeq n_s = 0.16$ fm^{-3}, corresponding to mass densities $\rho_s = n_s m_n \simeq 2.7 \cdot 10^{14}$ g cm^{-3}. They also have roughly equal numbers of neutrons and protons: $x = Z/A \simeq 1/2$.

$$E(Z,N) = E_{bulk}A + E_{surf}A^{2/3} + E_{Coul}A^{5/3} + \cdots$$

$$E_{bulk} \simeq -16 + \frac{K}{18}\left(1 - \frac{n}{n_s}\right)^2 + S_V(1-2x)^2 + \cdots$$

$K \approx 240$ MeV is incompressibility parameter, $S_V \approx 30n/n_s$ MeV is volume symmetry parameter.

$$E_{surf} \simeq 18 - S_S(1-2x)^2 \ \text{ MeV},$$

where $S_S \simeq 45$ MeV is the surface symmetry parameter.

$$E_{Coul} = \frac{3}{5}\frac{x^2e^2}{r_0} \simeq 0.75x^2 \ \text{ MeV},$$

where $r_0 = (4\pi n_s/3)^{-1/3} \simeq 1.12$ fm.

Consider an infinite nucleus. Saturation density is where energy per particle for a given composition is minimized:

$$\left.\frac{\partial E_{bulk}}{\partial n}\right|_x = \frac{P_{bulk}}{n^2} = 0.$$

This occurs where pressure vanishes, or $n = n_s$ if $x = 1/2$. If $x \neq 1/2$, then

$$\frac{P_{bulk}}{n^2} = -\frac{K}{9n_s}\left(1 - \frac{n}{n_s}\right) + \frac{\partial S_V}{\partial n}(1-2x)^2,$$

$$\frac{n}{n_s} \simeq 1 - \left.\frac{9nS_V'(1-2x)^2}{K}\right|_{n_s} \simeq 1 - 1.1(1-2x)^2.$$

Optimize energy per particle with respect to composition:

$$\left.\frac{\partial E_{bulk}}{\partial x}\right|_n = -\left(\mu_n - \mu_p\right) = -4S_V(1-2x) = 0,$$

that is, $x = 1/2$ for all n. However, we've neglected electrons. Optimum composition determined by:

$$\mu_n - \mu_p = \mu_e, \quad 4S_V(1-2x) = \hbar c\left(3\pi^2 nx\right)^{1/3}.$$

S_V is assumed to be $S_0 n/n_s$, so as $n \to 0$, $x \to 0$. But for $n \to \infty$,

$$n = \left(3\pi^2 x\right)^{2/3} \left(\frac{\hbar c n_s}{4S_0\left(1-2x\right)}\right)^{3/2}$$

where $S_0 = 30$ MeV. Thus $x \to 1/2$ in this limit. When $n = n_s, x \approx 0.04$.

A finite nucleus has an optimum mass, for a given x:

$$\left.\frac{\partial E/A}{\partial A}\right|_x = -\frac{E_{surf}}{3A^{4/3}} + \frac{2E_{Coul}}{3A^{1/3}} = 0.$$

This becomes $E_{surf}A^{2/3} = 2E_{Coul}A^{5/3}$, the so-called *Nuclear Virial Theorem.* So

$$A_{opt} = E_{surf}/\left(2E_{Coul}\right). \tag{1}$$

This increases with decreasing x near $1/2$ roughly as x^{-2}. For $x = 1/2$, $A_{opt} = 18/.375 \simeq 48$. A nucleus also has an optimum charge, for a given A:

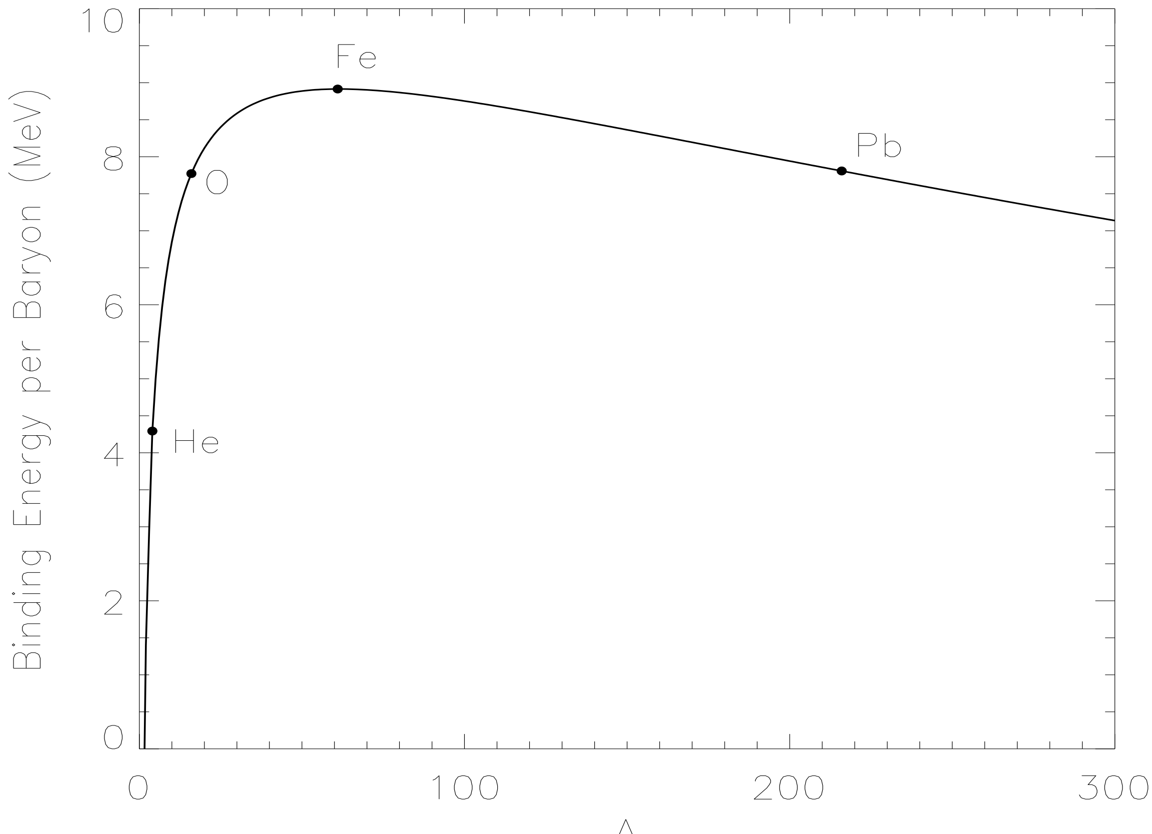
$$\left.\frac{\partial E/A}{\partial x}\right|_A = -4\left(S_V - S_S A^{-1/3}\right)(1-2x) + 2E_{Coul}A^{2/3}/x = 0,$$

$$x_{opt} = \left[2 + \frac{0.75A^{2/3}}{2\left(S_V - S_S A^{-1/3}\right)}\right]^{-1}. \tag{2}$$

This path represents the Valley of Beta Stability in the Chart of the Nuclides. If there was no Coulomb energy, or the mass is very small, then $x_{opt} = 1/2$. For larger masses, x_{opt} decreases from $1/2$. The simultaneous solution of Eqs. (1) and (2) yields $x_{opt} \simeq 0.432, A_{opt} \simeq 61, Z_{opt} \simeq 26$, or ^{61}Fe. Along the Valley of Beta Stability, the nuclear energy is

$$\begin{aligned} E/A =& -16 + 18A^{-1/3} + 0.375x_{opt}A^{2/3} \\ =& -16 + 18A^{-1/3} + \left[\frac{16}{3A^{2/3}} + \frac{1}{S_V - S_S A^{-1/3}}\right]^{-1}. \end{aligned}$$

This rises steeply with A to the maximum, then decreases relatively slowly beyond the maximum.



The binding energies of ^{4}He and ^{56}Fe, per baryon, are about 7.1 and 8.9 MeV, respectively, relative to individual neutrons and protons. The nuclear energy release in H burning is far greater than the energy released in all subsequent burning stages, which end in Fe formation.

Behavior of Nuclei at High Density

We treated nuclei in isolation. By the end of stellar evolution in massive stars, $\rho \simeq 10^7$ g cm^{-3}. The filling factor of nuclei is $u \simeq \rho/\rho_s \simeq 3.7 \cdot 10^{-8}$, so the distance between nuclei is about $2u^{-1/3} \approx 600$ nuclear radii. This is large enough that the effective nuclear Coulomb energy is diminished because of electron screening. For uniformly distributed electrons, the Coulomb energy becomes:

$$E_{Coul} = \frac{3}{5}\frac{x^2 e^2}{r_0}\left(1 - \frac{3}{2}u^{1/3} + \frac{u}{2}\right).$$

Therefore, at high densities, the average nuclear size increases as $(1 - \frac{3}{2}u^{1/3} + u/2)^{-1/3}$. However, decreasing proton fractions in nuclei lead to neutron drip.

Neutron Drip

The energies of the *last* neutron and proton in nuclei (also called the separation energy) are μ_n and μ_p:

$$\mu_n = \frac{\partial E_{nuc}}{\partial N}\Big|_Z = \frac{E_{nuc}}{A} - \frac{Z}{A}\frac{\partial\left(E_{nuc}/A\right)}{\partial\left(Z/A\right)}\Big|_A$$

$$\mu_p = \frac{\partial E_{nuc}}{\partial Z}\Big|_N = \frac{E_{nuc}}{A} + \left(1 - \frac{Z}{A}\right)\frac{\partial\left(E_{nuc}/A\right)}{\partial\left(Z/A\right)}\Big|_A.$$

So

$$\mu_n = -16 + 18A^{2/3} + \left(S_V - S_S A^{-1/3}\right)\left(1 - 4x^2\right) - E_{Coul}A^{2/3},$$

$$\mu_p = -16 + 18A^{2/3} + \left(S_V - S_S A^{-1/3}\right)(1-x)(2x-3) + \frac{(2-x)}{x}E_{Coul}A^{2/3}.$$

When $\mu_n > 0$, it is energetically favorable for neutrons to "drip" out of nuclei. This occurs around the density $4 \cdot 10^{11}$ g cm^{-3} when the nuclear proton fraction drops below about $x = 0.3$. For small proton fractions, the surface energy as given earlier becomes unphysically small. A better parametrization is

$$E_{surf} = \alpha\left[x^{-3} + (1-x)^{-3} + \beta\right]^{-1}$$

$$\alpha = 96E^2_{surf}\left(x = 1/2\right)/S_s, \qquad \beta = 96E_{surf}\left(x = 1/2\right)/S_s - 16.$$