Advanced Evolutionary Stages

Beyond helium burning, stellar evolution becomes increasingly dominated by neutrino processes: thermal pair creation and electron captures on protons in nuclei:

$$\begin{aligned} e^+ + e^- &\to \nu_e + \bar{\nu}_e \\ e^- + A_Z^N &\to \nu_e + A_{Z-1}^{N+1}. \end{aligned} \tag{1}$$

Temperatures and lifetimes of burning stages are determined by equilibrium between nuclear and neutrino rates.

Neutrino pair energy generation rate:

$$\dot{\epsilon}_{\nu\bar{\nu}} = n_{e^+} n_{e^-} \langle \sigma v E \rangle \tag{2}$$

where E is the total energy of the pair and the average is with respect to the $e^- - e^+$ velocity distribution. The annihilation cross section σ is of order G_F^2, proportional to energy squared, and the relevant velocity is c:

$$\sigma v = 1.42 \times 10^{-45} c \left[\left(\frac{w}{m_e c^2} \right)^2 - 1 \right] \text{ cm}^2, \tag{3}$$

where w is the center of momentum total energy. General result is complicated, but simplifies in NDNR and NDR limits.

NDNR ($T_9 < 1$). Recall that

$$n_{e^+} n_{e^-} = 2.3 \times 10^{58} T_9^3 e^{-11.9/T_9} \text{ cm}^{-6}. \tag{4}$$

Also, in the non-relativistic case $w \simeq E \simeq 2 m_e c^2$. Thus

$$\dot{\epsilon}_{\nu\bar{\nu}} = 4.9 \times 10^{18} T_9^3 e^{-11.9/T_9} \text{ erg cm}^{-3} \text{ s}^{-1}. \tag{5}$$

NDER ($T_9 > 3$). $e^- - e^+$ pairs are slightly degenerate even at high T. For $\mu_e \to 0$,

$$n_+ n_- = \frac{1}{\pi^4} \left(\frac{T}{\hbar c} \right)^6 F_2^2(0) \simeq 2.3 \times 10^{56} T_9^6 \text{ cm}^{-6}. \tag{6}$$

The average energy is $w \simeq 2[F_3(0)/F_2(0)]T$.

$$\dot{\epsilon}_{\nu\bar{\nu}} = 8.74 \times 10^{15} T_9^9 \ \mathrm{erg\ cm^{-3}\ s^{-1}}. \tag{7}$$

More accurate calculations gives 5.2 instead of 8.74.

Electron capture rates:

Depend sensitively on the Q-value (capture on free protons $\propto \Delta^5$, on heavy nuclei $\propto \Delta^{3-4}$, where $\Delta = \mu_e - \mu_n + \mu_p$ is available energy. For C or Ne burning, $\mu_e \simeq \mu_n - \mu_p$ and the matter is in beta equilibrium. But for Si burning and beyond, beta equilibrium is not maintained, $\Delta > 0$, and electron capture increases. Although ^{56}Fe has $Z/A \simeq 0.464$, $\langle Z/A \rangle = Y_e \simeq 0.42$ after Si burning. Because of thermal excitations, inverse processes (positron capture) can also proceed on some nuclei:

$$\begin{aligned} e^- + A_Z^N &\rightarrow \nu_e + A_{Z-1}^{N+1}, \\ e^+ + A_{Z-1}^{N+1} &\rightarrow \bar{\nu}_e + A_Z^N. \end{aligned} \tag{8}$$

This cyclic process occurs in spite of the fact that the available energy for one of the reactions is formally negative, because some of those nuclei are in excited states at finite temperature. The neutrino production rate from these so-called URCA processes is proportional to T^5, but their importance has to be considered on a case-by-case basis. Gamow named these processes URCA from a casino in Rio: you always lose (it also seems the name is similar to a Russian word for thief). For our purposes, it is sufficient to ignore electron capture and URCA processes until gravitational collapse itself sets in.

In general, there is a well defined sequence of stellar burning stages leading to an "onionskin-like" layering. T and the duration of the burning are found by equilibrium between neutrino and nuclear rates.

C burning:

$$\dot{\epsilon}_C \sim 7 \times 10^4 \left(\frac{X_C}{0.2}\right) \rho^2 T_9^{27} \ \text{erg cm}^{-3}\text{s}^{-1} \tag{9}$$

near $T_9 \sim 1$, where X_C is the C mass fraction. (Note that this rate is still uncertain to perhaps a factor of 2 or 3.) Equating Eqs. (9) and (5):

$$\rho^2 \left(\frac{X_C}{0.2}\right) T_9^{24} e^{11.9/T_9} \simeq 7 \times 10^{13} \ \text{g}^2\text{cm}^{-6}. \tag{10}$$

For $\rho \simeq 10^5$ g cm^{-3}, $T_9 \simeq 0.74$. This is an underestimate because of the extreme temperature sensitivity of the rates. Consider instead the total emission from a rate

$$\dot{\epsilon} = \dot{\epsilon}_o \left(\frac{\rho}{\rho_o}\right)^\lambda \left(\frac{T}{T_o}\right)^\nu \tag{11}$$

near ρ_o and T$_o$. In Lane-Emden variables, $\dot{\epsilon} = \dot{\epsilon}_o \theta^{n\lambda+\nu}$ and

$$\langle \dot{\epsilon} \rangle = \dot{\epsilon}_o \int_0^{\xi_1} \theta^{n\lambda+\nu} \xi^2 d\xi \Big/ \int_0^{\xi_1} \xi^2 d\xi. \tag{12}$$

n is the polytropic index. If $n\lambda + \nu >> 1$, the integrand is sharply peaked at the origin. Approximating $\theta \simeq e^{-\xi^2/6}$ (valid for all n) and extending the numerator's integral to ∞:

$$\langle \dot{\epsilon} \rangle = \frac{3\dot{\epsilon}_o}{\xi_1^2} \frac{\sqrt{27\pi/2}}{(n\lambda + \nu)^{3/2}}. \tag{13}$$

Assuming the core structure of massive stars is approximated by an $n = 3$ polytrope, the right hand side of Eq. (10) should be multiplied by $[(3\lambda+\nu)_C/(3\lambda+\nu)_\nu]^{3/2}$. We have $\lambda_C = 2, \nu_C = 27$, and for $T_9 \simeq 0.8$ $\lambda_\nu = 0, \nu_\nu = 3 + 11.9/.8 \simeq 18$. Thus the ratio is $(33/19)^{3/2} \simeq 2.5$ and the solution of Eq. (10) gives $T_9 = 0.82$.

The C burning duration is

$$\tau = \frac{\rho \Delta B}{\dot{\epsilon}_{nuc}} X_c \tag{14}$$

where $\Delta B = 4 \times 10^{17}$ erg g^{-1} is the specific energy released in C burning. Thus

$$\tau \simeq 10/T_9^{27} \ \text{yr} \simeq 2100 \ \text{yr}, \tag{15}$$

which is about 7 times too large compared to detailed calculations.

Following C burning are the Ne, O and Si burning stages. For the first two, $3 > T_9 > 1$ and neither the ER or NR limits apply. But the ER rate Eq. (7) is closer.

Ne burning:

This is a nuclear rearrangement 2Ne→ O + Mg. For $X_O = 0.7$

$$\begin{aligned} \dot{\epsilon} &\sim 2 \times 10^{24} \rho \left(\frac{X_{Ne}}{0.2} \right) T_9^{12} e^{-54.9/T_9} \ \text{erg cm}^{-3} \ \text{s}^{-1} \\ \Delta B &\simeq 1.1 \times 10^{17} \ \text{erg g}^{-1}. \end{aligned} \tag{16}$$

Note the single power of density, since this is a photodisintegration reaction. From Eq. (7), $\lambda_\nu = 0, \nu_\nu = 9$, and $\lambda_{Ne} = 1, \nu_{Ne} = 54.9/T_9 + 12$. We thus find the implicit equation for T_9:

$$3.8 \times 10^8 \rho T_9^3 e^{-54.9/T_9} \left(\frac{15 + 54.9/T_9}{9} \right)^{3/2} = 1, \tag{17}$$

which gives $T_9 \simeq 1.4$ if $\rho = 2 \times 10^5$ g cm^{-3}. The neon burning duration is about 17 yr, but will be increased by convective mixing.

O burning:

$2O^{16} \rightarrow {}^{28}$Si+He or ^{32} S via a large number of secondary reactions. In this stage almost all nuclei heavier than Fe photodisintegrate into Fe peak nuclei, and a general increase of the neutron excess $(N-Z)/A \simeq 0.01$ occurs. A good approximation for the energy generation is 2O→S

$$\dot{\epsilon} \sim 10^{-5} X_O^2 \rho^2 T_9^{33} \text{ erg cm}^{-3} \text{ s}^{-1}, \quad \Delta\text{B} \simeq 5 \times 10^{17} \text{ erg g}^{-1}. \tag{18}$$

Using $X_O \simeq 0.7, \rho \simeq 2 \times 10^6$ g cm^{-3} we find

$$3.8 \times 10^{-9} T_9^{24} \, (39/9)^{3/2} = 1 \tag{19}$$

or $T_9 \simeq 2.0$. The indicated lifetime is about 2 months.

Si burning:

Essentially photodisintegraton rearrangement into Fe-peak nuclei. Si fusions are prohibited by the high Coulomb barrier. α particles released by photodisintegrations are added to heavy nuclei, pushing the distribution toward the optimum binding state, or nuclear statistical equilibrium. Only the density, temperature and neutron excess or Y_e are needed to find nuclear abundances (if binding energies and partition functions are known). Energy generation is

$$\begin{aligned} \dot{\epsilon} \sim & 4 \times 10^{27} X_{Si} \rho T_9^{6.5} e^{-142.1/T_9} \\ \simeq & 3.2 \times 10^{13} X_{Si} \rho \left(\frac{T_9}{3.5}\right)^{47} \text{ erg cm}^{-3}\text{s}^{-1}, \\ \Delta B \simeq & 1.9 \times 10^{17} \text{ erg g}^{-1} \end{aligned} \tag{20}$$

near $T_9 = 3.5$. The implicit T equation is

$$7.8 \times 10^{-8} X_{Si} \rho \left(\frac{T_9}{3.5}\right)^{38} \left(\frac{50}{9}\right)^{3/2} = 1 \tag{21}$$

or $T_9 \simeq 3.3$ if $\rho \simeq 10^7$ g cm^{-3}. The indicated lifetime is about 1.5 days.

Nuclear Statistical Equilibrium

The free energy per baryon of a nucleus (Z, A) is

$$F(Z,A) = B(Z,A) + \frac{T}{A}\left(\ln\left[\left(\frac{2\pi\hbar^2}{mAT}\right)^{3/2}\frac{n(Z,A)}{G(Z,A)}\right] - 1\right) \tag{22}$$

where $B(Z, A), G(Z, A)$, and $n(Z, A)$ are the binding energy per particle, partition function, and number density of the nucleus, respectively. Nuclei and nucleons are treated as non-degenerate and non- interacting. The baryon mass is m. The total free energy density is then

$$f = \sum_{Z,A} F(Z,A)\, n(Z,A)\,. \tag{23}$$

Two constraints, mass and charge conservation:

$$\sum_{Z,A} A n(Z,A) = n \qquad \sum_{Z,A} Z n(Z,A) = nY_e. \tag{24}$$

Minimizing f with respect to each $n(Z, A)$, subject to these constraints, gives

$$F(Z,A) + T/A - \lambda_1 A - \lambda_2 Z = 0. \tag{25}$$

The total free energy density is therefore

$$f = -P + \mu_n n(1 - Y_e) + \mu_p n Y_e = -\sum_{Z,A}\frac{n(Z,A)\,T}{A} + \lambda_1 n + \lambda_2 n Y_e. \tag{26}$$

Thus, $\lambda_1 = \mu_n$ and $\lambda_2 = -(\mu_n - \mu_p) \equiv -\hat{\mu}$. The number densities of each nucleus are then

$$n(Z,A) = G(Z,A)\left(\frac{mAT}{2\pi\hbar^2}\right)^{3/2}\exp\left[-\frac{AB(Z,A) + N\mu_n + Z\mu_p}{T}\right]. \tag{27}$$

Number densities of nucleons are

$$n_{n,p} = 2\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} e^{\mu_{n,p}/T}, \tag{28}$$

so

$$n\left(Z,A\right) = G\left(\frac{mT}{2\pi\hbar^2}\right)^{\frac{3(1-A)}{2}}\left(\frac{1}{2}\right)^A A^{3/2} n_n^{A-Z} n_p^Z e^{-\frac{AB(Z,A)}{T}}. \tag{29}$$

At low temperatures, $G(Z,A) \simeq 1$, but above a few MeV,

$$G \simeq \frac{\pi}{6aT} e^{aT} \tag{30}$$

where $a \simeq A/9$ MeV^{-1} is the usual level density parameter. The partition function will begin to dominate the composition when $aT > AB/T$, or when $T > \sqrt{9B} \simeq 9$ MeV. Nuclei are dissociated by such large T, except near nuclear density.

The optimum nucleus satisfies $\partial n/\partial Z|_A = \partial n/\partial A|_Z = 0$, or

$$\hat{\mu} = A\left(\left.\frac{\partial B}{\partial N}\right|_Z - \left.\frac{\partial B}{\partial Z}\right|_N\right) = -\left.\frac{\partial B}{\partial\left(Z/A\right)}\right|_A \tag{31}$$

using Eq. (30). With electronic energy also minimized, we have $\hat{\mu} = \mu_e$, the usual condition for beta equilibrium. For densities above 10^6 g cm^{-3}, the electrons are relativistic. In stellar cores, the electrons are degenerate as well. Eq. (23) should properly include the rest masses of the neutrons and protons, so an extra term $(m_n - m_p)c^2 Y_e$ appears:

$$\hat{\mu} + \left(m_n - m_p\right)c^2 = \mu_e.$$

Since $Y_e = \langle Z/A\rangle$, we see that the most abundant nucleus is the one that has the largest binding energy *for a given neutron excess*. Thus, although ^{62}Ni has a larger binding energy than ^{56}Ni (8.795 MeV compared to 8.790 MeV), ^{56}Ni is more abundant near the valley of beta stability.