# Implicit Integration – Henyey Method

In realistic stellar evolution codes, instead of a direct integration using, for example, the Runge-Kutta method, one employs an iterative implicit technique. This is because the structure equations have to be solved in parallel with the energy transport equations. If timesteps are chosen to be small enough, convergence is very rapid. It is convenient to reformulate the original structure equations as

$$\frac{d\ln r}{dm} = \frac{1}{4\pi\rho r^3};$$

$$\frac{d\ln P}{dm} = -\frac{Gm}{4\pi P r^4}.$$

In order to reduce the dynamic range of the variables, we definee $x = \ln r$, $y = \ln P$ and $q = \ln\rho$ and input an equation of state $\rho(P)$ or equivalently $q(y)$. We rewrite these differential equations as finite-difference equations to be zeroed at each position $i$:

$$\phi_i = y_i - y_{i-1} + \frac{G}{8\pi}\left(m_i^2 - m_{i-1}^2\right)\mathrm{e}^{-\frac{1}{2}(y_i+y_{i-1})-2(x_i+x_{i-1})},$$

$$\psi_i = x_i - x_{i-1} - \frac{1}{4\pi}\left(m_i - m_{i-1}\right)\mathrm{e}^{-\frac{3}{2}(x_i+x_{i-1})-\frac{1}{2}(q_i+q_{i-1})}.$$

These equations are valid for $2 \le i \le N-1$, where $i$ is the zone number and $N$ is the number of (radial) zones into which we divide the star. Thus, $m_i, y_i, x_i$ are the values of the respective variables at the outer edge of the $i$th zone. Note that the values of $m_i$ are set in advance for the star and will not change during the iteration. Also note how the finite differencing is done so as to reduce errors:

$$mdm = \frac{1}{2}dm^2 \to \frac{1}{2}\left(m_i^2 - m_{i-1}^2\right), \quad \frac{1}{P} = \mathrm{e}^{-\ln P} \to \mathrm{e}^{-\frac{1}{2}(y_i+y_{i-1})}.$$

At the inner and outer boundaries, these equations must be rewritten since at $i = 0$, $x \to -\infty$, and at $i = N$, $y \to -\infty$. The inner boundary can be approximated using the incompressible fluid result

$$P(r) \simeq P_c - \frac{G}{2}\left(\frac{4\pi}{3}\rho_c^4 m(r)^2\right)^{1/3}; \qquad r \simeq \left(\frac{3m(r)}{4\pi\rho_c}\right)^{1/3},$$

which are valid near the origin. Thus, using the subscript $_0$ for the origin,

$$\phi_1 = y_1 - y_0 + \frac{G}{2}\left(\frac{4\pi}{3}\right)^{1/3} M_1^{2/3} e^{4q_0/3 - y_0}\,;$$

$$\psi_1 = x_1 - \frac{1}{3}\left(\ln\left[\frac{3M_1}{4\pi}\right] - q_0\right)\,.$$

The surface can be approximated in several ways. For example, the polytropic index might be nearly constant there, with a value $\gamma_R$, and the mass in the outermost zone is negligible compared to the total mass $M$. (In fact, we will assume $\gamma_R = 4/3$.) Then it is easy to show that two independent equations for the behavior of P and r near the surface are

$$P = \frac{GM\left(M - m\left(r\right)\right)}{4\pi r^4},$$

$$\frac{P}{\rho} = GM\left(1 - \gamma_R^{-1}\right)\left(\frac{1}{r} - \frac{1}{R}\right).$$

These lead to

$$\phi_N = y_{N-1} + 2\left(x_N + x_{N-1}\right) - \ln\frac{Gm_N\left(m_N - m_{N-1}\right)}{4\pi}\,;$$

$$\psi_N = e^{y_{N-1} - q_{N-1}} - Gm_N\left(1 - \left.\frac{dq}{dy}\right|_{q\to 0}\right)\left(e^{-x_{N-1}} - e^{-x_N}\right)\,.$$

Note that $\left.\frac{dq}{dy}\right|_{q\to 0} = \gamma_R$. Thus, in total, there are $N$ values of $x_i$ and $y_i$ to solve for, and we have $N$ equations each for $\phi$ and $\psi$ to do it with.

Since we want to solve $\phi_i(x_i, x_{i-1}, y_i, y_{i-1}) = 0$ and $\psi_i(x_i, x_{i-1}, y_i, y_{i-1}) = 0$, we expand them in Taylor series:

$$\phi_i + a_i \Delta x_{i-1} + b_i \Delta y_{i-1} + c_i \Delta x_i + d_i \Delta y_i = 0\,;$$

$$\psi_i + a'_i \Delta x_{i-1} + b'_i \Delta y_{i-1} + c'_i \Delta x_i + d'_i \Delta y_i = 0\,,$$

where the notation $\Delta x_i$ and $\Delta y_i$ refers to the changes in the values of $x_i$ and $y_i$ that will zero the $\phi$ and $\psi$ equations. That is, we need to solve the above equations for these $\Delta$'s in order to determine how much the $x$'s and $y$'s should be changed for each iteration. The quantities $a, b, c$ and $d$ are the derivatives

$$a_i = \frac{\partial \phi_i}{\partial x_{i-1}}, \qquad b_i = \frac{\partial \phi_i}{\partial y_{i-1}}, \qquad c_i = \frac{\partial \phi_i}{\partial x_i}, \qquad d_i = \frac{\partial \phi_i}{\partial y_i}\,,$$

$$a'_i = \frac{\partial \psi_i}{\partial x_{i-1}}, \qquad b'_i = \frac{\partial \psi_i}{\partial y_{i-1}}, \qquad c'_i = \frac{\partial \psi_i}{\partial x_i}, \qquad d'_i = \frac{\partial \psi_i}{\partial y_i}\,.$$

These are functions of the $x$'s and $y$'s. These equations are linear in the $\Delta$'s, so we can assume

$$\Delta x_i = -\gamma_i - \alpha_i \Delta y_i \, ; \qquad \Delta x_{i-1} = -\gamma_{i-1} - \alpha_{i-1} \Delta y_{i-1} \, .$$

By substitution and elimination into the equations for $\phi$ and $\psi$, we find

$$\gamma_i = \frac{\left(b_i' - a_i' \alpha_{i-1}\right)\left(\phi_i - a_i \gamma_{i-1}\right) - \left(b_i - a_i \alpha_{i-1}\right)\left(\psi_i - a_i' \gamma_{i-1}\right)}{c_i \left(b_i' - a_i' \alpha_{i-1}\right) - c_i' \left(b_i - a_i \alpha_{i-1}\right)} \, ;$$

$$\alpha_i = \frac{d_i \left(b_i' - a_i' \alpha_{i-1}\right) - d_i' \left(b_i - a_i \alpha_{i-1}\right)}{c_i \left(b_i' - a_i' \alpha_{i-1}\right) - c_i' \left(b_i - a_i \alpha_{i-1}\right)} \, .$$

We also can find

$$\Delta y_{i-1} = -\frac{\left(\psi_i - a_i' \gamma_{i-1} + c_i' \Delta x_i + d_i' \Delta y_i\right)}{b_i' - a_i' \alpha_{i-1}} .$$

Now we are in a position to determine new guesses from the original ones. Note that $r_0 = 0$ implies $\Delta x_0 = 0$ since the radius at the origin is always zero. Thus we must have $\gamma_0 = \alpha_0 = 0$. We can loop through the above equations for $\gamma$ and $\alpha$ to now find $\alpha_i$ and $\gamma_i$ from their values for $i-1$. From the fact that the pressure vanishes on the outer boundary, $\Delta y_N = 0$ which also implies $\Delta x_N = -\gamma_N$. We can find $\Delta y_{N-1}$ in terms of $\Delta x_N, \Delta y_N$ and the coefficients $a_{N-1}', b_{N-1}', c_{N-1}', d_{N-1}'$, and then employ $\Delta x_{N-1} = -\gamma_{N-1} - \alpha_{N-1} \Delta y_{N-1}$. In this way, one can loop back to find the remaining $\Delta y$'s and $\Delta x$'s. Note that this is a form of Gaussian elimination.

When the changes $\Delta x_i$ and $\Delta y_i$ become small enough, we have convergence. It is important to note that this is a Newton-Raphson technique, and therefore its success depends upon suitable initial guesses. I have found that an initial guess based upon the analytic solution for an incompressible gas works adequately. For the incompressible gas, we have

$$m\,(r) = \frac{4\pi \rho_c r^3}{3} \, ; \qquad P\,(r) = P_c - \frac{2\pi}{3} G \rho^2 r^2 \, .$$

These can be expressed also as

$$r\,(m) = \left(\frac{3m}{4\pi \rho_c}\right)^{1/3} \, ; \qquad P\,(m) = P_c - G \frac{2\pi}{3} \left(\frac{3m\rho_c^2}{4\pi}\right)^{2/3} \, .$$

The values of $P_c$ and $\rho_c$ in this approximation are found from

$$P_c / \rho_c^{4/3} = \left(2\pi G/3\right)\left(.75 M/\pi\right)^{2/3} ,$$

which follows from $P(m = M) = 0$, combined with the equation of state $P_c(\rho_c)$.

# Henyey for Relativistic Stars

To include the effects of General Relativity, one must distinguish between the gravitational mass $m(r)$ and the baryon mass $b(r)$, where $b(r)$ is the number of baryons within a radius $r$ times the baryon mass ($m_B$). Because in GR the gravitational mass is dependent upon the local gravitational field, but the baryon number is an invariant quantity, we must use $b(r)$ as the independent variable instead of $m(r)$. The relevant equations become

$$\begin{aligned}\frac{d\ln r}{db} &= \frac{\sqrt{1-2Gm/rc^2}}{4\pi n m_B r^3}\\ \frac{d\ln P}{db} &= -\frac{G\left(m+4\pi r^3 P/c^2\right)\left(\rho+P/c^2\right)}{4\pi r^4 n m_B P\sqrt{1-2Gm/rc^2}}\\ \frac{dm}{db} &= \frac{\rho}{n m_B}\sqrt{1-2Gm/rc^2}.\end{aligned} \tag{1}$$

Here, the total mass density is $\rho = n(m_B + e/c^2)$ where $n$ is the baryon density and $e$ is the internal energy per baryon. Employing $y = \ln(P/c^2), x = \ln r$ and $q = \ln\rho$, with, in addition, $z = \ln(nm_B)$, we find

$$\begin{aligned}\phi_i =& y_i - y_{i-1} + \frac{G}{4\pi c^2}\frac{b_i-b_{i-1}}{\Lambda}\left[1+e^{\frac{1}{2}(q_i+q_{i-1}-y_i-y_{i-1})}\right]\\ &\left(\frac{m_i+m_{i-1}}{2}+4\pi e^{\frac{3}{2}(x_i+x_{i-1})+\frac{1}{2}(y_i+y_{i-1})}\right)e^{-\frac{1}{2}(z_i+z_{i-1})-2(x_i+x_{i-1})}\\ \psi_i =& x_i - x_{i-1} - \frac{1}{4\pi}\left(b_i-b_{i-1}\right)e^{-\frac{3}{2}(x_i+x_{i-1})-\frac{1}{2}(z_i+z_{i-1})}\Lambda,\\ \chi_i =& m_i - m_{i-1} - \left(b_i-b_{i-1}\right)e^{\frac{1}{2}(q_i+q_{i-1}-z_i-z_{i-1})}\Lambda,\end{aligned} \tag{2}$$

where

$$\Lambda = \sqrt{1-\frac{G}{c^2}\left(m_i+m_{i-1}\right)e^{-\frac{1}{2}(x_i+x_{i-1})}}.$$

At the inner and outer boundaries, the first two equations must be replaced by equations similar to before, but the third equation is well-behaved at these boundaries and does not have to be replaced. Thus, at the inner boundary,

$$
\begin{aligned}
\phi_1 =& y_1 - y_0 + \frac{G}{c^2}\left(\frac{\pi m_1^2}{6}\right)^{1/3} e^{\frac{4}{3}q_0 - y_0}\left(1+e^{y_0-q_0}\right)\left(1+3e^{y_0-q_0}\right), \\
\psi_1 =& x_1 - \frac{1}{3}\left[\ln\left(\frac{3m_1}{4\pi}\right) - q_0\right], \\
\chi_1 =& m_1 - b_1 e^{\frac{1}{2}(q_1+q_0-z_1-z_0)}\sqrt{1-\frac{2G}{c^2}\left(\frac{\pi m_1^2}{3}\right)^{1/3} e^{q_0/3}},
\end{aligned} \tag{3}
$$

and, at the outer boundary,

$$
\begin{aligned}
\phi_N =& y_{N-1} + 2\left(x_N + x_{N-1}\right) - \ln\left[\frac{G\left(m_N+m_{N-1}\right)\left(b_N-b_{N-1}\right)}{8\pi\sqrt{1-2Gm_N e^{-x_N}/c^2}}\right], \\
\psi_N =& e^{y_{N-1}-z_{N-1}} + \frac{1}{2}\left(1-\left.\frac{dz}{dy}\right|_{y\to 0}\right)\ln\left[\frac{1-2Gm_N e^{-x_{N-1}}/c^2}{1-2Gm_N e^{-x_N}/c^2}\right], \\
\chi_N =& m_N - m_{N-1} - \left(b_N - b_{N-1}\right)\sqrt{1-2Gm_N e^{-x_N}/c^2}.
\end{aligned} \tag{4}
$$

To implement the boundary conditions, it is convenient to use the linear relation

$$
\Delta y_i = -\gamma_i - \alpha_i \Delta x_i - \beta_i \Delta m_i, \tag{5}
$$

and the corresponding expression for $i-1$. At the inner boundary, we must have $\Delta x_0 = \Delta m_0 = 0$, so $\Delta y_0 = -\gamma_0$. Similarly, at the outer boundary, the condition $\Delta y_N = 0$ implies that $\gamma_N = \alpha_N = \beta_N = 0$. Therefore, we seek relations for $\gamma_{i-1}, \alpha_{i-1}$ and $\beta_{i-1}$ in terms of $\gamma_i, \alpha_i$ and $\beta_i$. In addition, we need expressions for $\Delta y_i$, $\Delta x_i$ and $\Delta m_i$ in terms of $\Delta y_{i-1}, \Delta x_{i-1}$ and $\Delta m_{i-1}$. Therefore the recursions will proceed oppositely to the scheme we employed for the Newtonian calculations.

We expand the functions $\phi_i, \psi_i, \chi_i$ in Taylor series in the variables $y_{i-1}, y_i$, $x_{i-1}, x_i$, $m_{i-1}, m_i$, which will define the coefficients $a_i, a_i', a_i''$ and so forth for $b, c, d, e$ and $f$:

$$
\begin{aligned}
&\phi_i + a_i\Delta y_{i-1} + b_i\Delta x_{i-1} + c_i\Delta m_{i-1} + d_i\Delta y_i + e_i\Delta x_i + f_i\Delta m_i = 0, \\
&\psi_i + a_i'\Delta y_{i-1} + b_i'\Delta x_{i-1} + c_i'\Delta m_{i-1} + d_i'\Delta y_i + e_i'\Delta x_i + f_i'\Delta m_i = 0, \\
&\chi_i + a_i''\Delta y_{i-1} + b_i''\Delta x_{i-1} + c_i''\Delta m_{i-1} + d_i''\Delta y_i + e_i''\Delta x_i + f_i''\Delta m_i = 0.
\end{aligned} \tag{6}
$$

We assume the linear relation Eq. (5) exists among the Δ's. One finds the relations

$$
\begin{aligned}
\gamma_{i-1} =& \frac{B'\Psi - B\Phi + B''X + \gamma_i \left[b_i''A'' + b_iA - b_i'A'\right]}{B'D' - BD + B''D''}, \\
\alpha_{i-1} =& \frac{B'E' - BE + B''E''}{B'D' - BD + B''D''}, \\
\beta_{i-1} =& \frac{B'F' - BF + B''F''}{B'D' - BD + B''D''}, \\
\Delta x_i =& \frac{\gamma_i (A - \Phi) - \Delta y_{i-1}D - \Delta x_{i-1}E - \Delta m_{i-1}F}{C''B' - C'B''}, \\
\Delta m_i =& \frac{\psi_i B - \phi_i B' + \gamma_i \left(a_ib_i' - a_i'b_i\right) - \Delta y_{i-1}G - \Delta x_{i-1}G' - \Delta m_{i-1}G''}{c_iB' - c_i'B + \beta_i \left(a_i'b_i - a_ib_i'\right)},
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
\Phi =& \psi_iC'' - \chi_iC', \quad \Psi = \phi_iC'' - \chi_iC, \quad X = \psi_iC - \phi_iC', \\
A =& a_i'C'' - a_i''C', \quad A' = a_iC'' - Ca_i'', \quad A'' = a_iC' - a_i'C, \\
B =& b_i - \alpha_ia_i, \quad B' = b_i' - \alpha_ia_i', \quad B'' = b_i'' - \alpha_ia_i'', \\
C =& c_i - \beta_ia_i, \quad C' = c_i' - \beta_ia_i', \quad C'' = c_i'' - \beta_ia_i'', \\
D =& d_i'C'' - d_i''C', \quad D' = d_iC'' - d_i''C, \quad D'' = d_iC' - d_i'C, \\
E =& e_i'C'' - e_i''C', \quad E' = e_iC'' - e_i''C, \quad E'' = e_iC' - e_i'C, \\
F =& f_i'C'' - f_i''C', \quad F' = f_iC'' - f_i''C, \quad F'' = f_iC' - f_i'C. \\
G =& d_iB' - d_i'B, \quad G' = e_iB' - e_i'B, \quad G'' = f_iB' - f_i'B.
\end{aligned}
\tag{8}
$$

These are supplemented by Eq. (5).