Homework # 4Due Mar 22
Gravitational Hydrostatics Part II

In the case of a more general equation of state, the scaling properties of a polytrope do not hold. Reverting to the original hydrostatic equations in the variables P, ρ, m and r, we may again use the Runge-Kutta method using P and m as the dependent variables and r as the independent variable. We must use the equation of state $\rho(P)$ also. With explicit integration, we begin at the origin $m = r = 0$ assuming a given value of ρ_c or P_c. We integrate until $P = 0$ is reached as above, thus determining R and M. In this general case, we have the two problems previously identified to be wary of (i.e., 0/0 behavior near the origin, and negative densities or pressures if too large a stepsize is used near the outer boundary) and in addition one has to take care of units, being careful such that quantities large compared to 1 or small compared to 1 do not occur.

These difficulties can mostly be eliminated by judicious choices of variables. In my experience, it is useful to use $x \equiv r^2$ instead of r and the enthalpy h, where $dh = dP/\rho$, in lieu of both P and ρ, and to invert the hydrostatic equations. Thus, we find

$$\frac{dx}{dh} = -\frac{2x^{3/2}}{Gm},$$
$$\frac{dm}{dh} = -\frac{4\pi x^2 \rho(h)}{Gm}.$$

Note that both equations are finite at the outer boundary, and, since $m \to 4\pi\rho x^{3/2}/3$ when $x \to 0$, both equations are also finite at the center: $dx/dh|_{x\to 0} = -3/(2\pi\rho_c G)$; $dm/dh|_{x\to 0} = -3x^{1/2}/G = 0$. The integration is performed over the range $h_c \leq h \leq 0$ where h_c is the central value of h. Thus the grid size could be set to $h_c/100$, for example, and no Newton-Raphson is needed to limit the stepsize near the outer boundary.

However, the usual case is to find the structure of a star of a given mass without knowing the correct value of the central density or h_c in advance. One has to guess this value, determine the mass by integration of the hydrostatic equations, change the central density or h_c, re-evaluate the mass, and iterate until the calculated mass equals the required mass. We will employ Newton-Raphson iteration to do this.

The following is a brief description of the Newton-Raphson technique for zeroing a one-dimensional function for which the functional derivative is complicated or non-analytic. This occurs many times.

Suppose we want to solve the equation $f(x) = 0$. An initial guess for the root, x_0, unless it is magically correct, will lead to a non-zero f: $f(x_0) \neq 0$. Now expand the function f in a Taylor series to find the value at a nearby point, say x_1:

$$f(x_1) \simeq f(x_0) + (df/dx)_{x_0} (x_1 - x_0).$$

Setting $f(x_1) = 0$ will establish a value for x_1, giving an estimate of a stepsize:

$$x_1 - x_0 = -f(x_0) / (df/dx)_{x_0}.$$

Unless $f(x)$ is a linear function, however, $f(x_1)$ will not turn out to be 0, but we can repeat this procedure, so that at the ith step we have

$$x_i - x_{i-1} = -f(x_{i-1}) / (df/dx)_{x_{i-1}}.$$

If the function $f(x)$ is reasonably well-behaved, and if x_0 is not too far from its root, the method will converge after a few iterations. When $|x_i - x_{i-1}|/x_i$ is sufficiently small (say, $< 10^{-6}$) we can claim convergence. However, if multiple roots exist or if the function has a derivative that changes sign in the vicinity of the root, this technique can fail and another approach, such as bisection, will have to be used. Thankfully, stellar structure equations are generally well-behaved.

To implement this scheme in the stellar structure case, however, it is necessary to know the derivative $f' = df/dx$ (i.e., in our case, dM/dh_c). This is a non-analytic function. We can, however, determine it numerically with almost no additional computation and implement the numerical derivative into the Newton-Raphson scheme. To begin, choose an estimate for the zero, x_0, and determine $f_0 = f(x_0)$. Then arbitrarily choose $x_{-1} = 0.999x_0$ and evaluate $f_{-1} = f(x_{-1})$. Then, f_0' can be estimated as

$$f_0' = (f_0 - f_{-1}) / (x_0 - x_{-1}).$$

Now we can begin to iterate. The next guess for the zero is

$$x_1 = x_0 - f_0/f_0' = x_0 - f_0 (x_0 - x_{-1}) / (f_0 - f_{-1}).$$

Next. we can determine $f_1 = f(x_1)$, and then automatically have an estimate for the derivative there, $f_1' = (f_1 - f_0)/(x_1 - x_0)$. Therefore, the iterative scheme at the ith stage is

$$x_i = x_{i-1} - \frac{f_{i-1}}{f_{i-1}'} = x_{i-1} - f_{i-1}\frac{x_{i-1} - x_{i-2}}{f_{i-1} - f_{i-2}} = \frac{f_{i-1}x_{i-2} - f_{i-2}x_{i-1}}{f_{i-1} - f_{i-2}},$$

where we used
$$f_i' = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}.$$
Convergence occurs either when $x_i - x_{i-1}$ or f_i becomes very small.

The homework assignment is:

Combine the Runge-Kutta and Newton-Raphson methods to solve for the structure of a star of mass $M = 1$ M$_\odot$ for which the equation of state is

$$P = K\rho^2 \left(1 + \rho/\rho_0\right).$$

Use $\rho_0 \simeq 4$ gm cm^{-3} and $K \simeq 10^{14}$ erg cm^3 g^{-2}. This equation of state does give an analytic solution for $h(r)$ or $\rho(r)$ which allows you to check your numerical results. I want to see your code and graphs of $r(h)$, $m(h)$, $\rho(r)$ and $P(r)$. What is the radius of your star?