Homework #2 Due Feb 24
Gravitational Hydrostatics

In this exercise, you will consider the application of the polytropic equation of state $P = K\rho^\gamma$ to hydrostatic equilibrium.

The equations governing Newtonian hydrostatic equilibrium are

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}; \qquad \frac{dm(r)}{dr} = 4\pi\rho(r)r^2$$

Let's express this in terms of dimensionless variables:

$$r = A\xi, \quad \theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}, \quad A = \left[(n+1)K\rho_c^{1/n-1}/(4\pi G)\right]^{1/2},$$

where ρ_c is the central density. Then the Lane-Emden equation is found:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{1}$$

Boundary conditions for Eq. (1) are θ=1 and $\theta' = d\theta/d\xi = 0$ at ξ=0. The outer extent of the polytrope is determined by the value of ξ, ξ_1, at which θ has its first zero. The only parameter, for these boundary conditions, is n. Each value of n gives rise to a different structure. Changes in other quantities, such as ρ_c, G, or K, only affect the scale.

Explicit Solution of Differential Equations

There are two basic ways in which the Lane-Emden equation may be solved. The first is to use an **explicit** scheme, which is a direct integration, or shooting method. Write the equations as

$$\theta' = y,$$
$$y' = -\frac{2}{\xi}y - \theta^n.$$

Beginning from the origin $\xi = 0$, where $\theta = 1$ and $y = 0$, one integrates until $\theta = 0$ is reached. This point, ξ_1, is the outer edge of the configuration. We want to know also the value of y_1 there. The simplest scheme is to sequentially advance the variables using a finite-differencing technique. Let $d\xi = \xi_{i+1} - \xi_i$ be a fixed interval size, say 0.1, where $i = 1$ at the center of the star (at this stage we don't know how many intervals to use). We write

$$d\theta = \theta_{i+1} - \theta_i = y d\xi \simeq y_i(\xi_{i+1} - \xi_i),$$
$$dy = y_{i+1} - y_i = -\left(\frac{2y}{\xi} + \theta^n\right)d\xi \simeq -\left(\frac{2y_i}{\xi_i} + \theta_i^n\right)(\xi_{i+1} - \xi_i),$$

where we use only known values on the right-hand sides of these equations (this is known as explicit differencing). Obviously, $\theta_1 = 1$ and $y_1 = 0$, and $\xi_1 = 0$ and $\xi_2 = d\xi$. The procedure is then to sequentially step through by calculating θ_2 and y_2, then θ_3 and y_3.

There are two aspects of this problem that require care.

1.) At the origin $\xi = 0$ the second of the above equations is apparently singular. Actually, it is not, because $y \to 0$ "faster" than ξ does. However, the computer doesn't know this, and it is necessary to expand $\theta(\xi)$ near the origin in a power series to find the $\xi = 0$ limit of θ'/ξ. Demonstrate that for *all* indices n, the series expansion valid near the origin is

$$\theta = 1 - \xi^2/6 + \cdots.$$

One sees, therefore, that $y/\xi = -1/3$ near the origin. Thus, $y_1/\xi_1 = -1/3$.

2.) In all explicit schemes, one takes steps of finite size $d\xi$. Near the outer boundary, however, one must take care that a step does not result in $\theta < 0$, since θ^n is needed. The code will croak if it has to take a negative number to a non-integer power. Show that $\theta'' = y'$ is positive near $\theta = 0$ for all n. Then show that an estimate of the next stepsize guaranteed not to be so large that $\theta < 0$ within the next step is given by $\delta\xi = -\theta/\theta'$. Therefore, after each step is taken, one needs to check that $-\theta/y < d\xi$, and if this is not true, redefine the next stepsize such that $d\xi = -\theta/y$. This allows one to gradually approach the boundary without ever exceeding it. You should continue the integration until θ becomes quite small, like 10^{-10} (you will never reach exactly 0). Once the values ξ and θ' are found at the outer boundary, one may deduce R and M using the appropriate transformation equations. Incidentally, the method of choosing stepsizes near the boundary is a specific example of *Newton-Raphson iteration.*

A much more accurate method than the simple stepping method is provided by the *Runge-Kutta method.* To solve the Lane-Emden equation, you should use the 4th order scheme described in *Numerical Recipes.*

The first numerical project is to solve the Lane-Emden equation for a polytrope. Do the cases $n = 0, n = 1$ and $n = 2$. Show plots of $\theta(\xi)$ for these cases.