Stellar Birth

Criterion for stability – neglect magnetic fields. Critical mass is called the Jean's mass M_J, where a cloud's potential energy equals its kinetic energy. If spherical with uniform temperature,

$$\Omega = GM_J^2/R \simeq M_J v_s^2/2 \tag{1}$$

or

$$M_J \simeq \frac{v_s^2}{2G}R = \left(\frac{5v_s^2}{6G}\right)^{3/2}\left(\frac{3}{4\pi\bar{\rho}}\right)^{1/2}. \tag{2}$$

$v_s \simeq \sqrt{2N_0T}$ is the sound velocity, $v_s \simeq 1$ km/s for $T = 100$ K. For a mean density $\bar{\rho} = 10^{-24}$ g cm^{-3}, $M_J \simeq 2 \cdot 10^4$ M$_\odot$. Eventual star formation must occur by fragmentation of more massive collapsing clouds.

Apply virial theorem to a cloud bounded by the ISM with pressure P_o.

$$\int_0^M \frac{Gm(r)}{r}dm = 3\int_0^M \frac{P}{\rho}dm - 4\pi R^3 P_o\,.$$

For an isothermal gas, using the fact that $m \propto r$, this is

$$\frac{GM^2}{R} = 3N_oTM - 4\pi R^3 P_o\,, \qquad P_o = \frac{3N_oTM}{4\pi R^3} - \frac{GM^2}{4\pi R^4}\,.$$

For small R, $P_o < 0$. For large R, $P_o > 0$ but tends to zero. There is a maximum of P_o, at R_m, which represents a stability limit. Reducing R from R_m lowers P_o and the ISM will force compression of the cloud, triggering collapse. This won't happen if $R > R_m$. From $(\partial P_o/\partial R)_M = 0$ one finds

$$R_m = \frac{4GM_J}{9N_oT}, \qquad M_J = \frac{9N_0T}{4G}R_m$$

where we identify the mass with the Jean's mass. Note that M_J is 4.5 times larger than the preceding estimate.

Clouds exceeding the Jeans mass are stabilized by magnetic fields.

$$\frac{B^2}{8\pi} \geq \frac{3GM^2}{4\pi R^4}, \tag{3}$$

$$M \leq BR^2/\sqrt{6G} \approx 50\,(R/\mathrm{pc})^2\,M_\odot,$$

if $B \approx 30\ \mu$G. A solar mass originates from within a region of less than about 0.14 pc.

Another source of support is rotation. For an Oort constant of 16 km/s/kpc, the angular rotation rate is $2 \cdot 10^{-16}$ s^{-1}. The Keplerian frequency is $\Omega_K = \sqrt{GM/R^3}$, implying a limit

$$M \geq \Omega_K^2 R^3/G = 0.8\,(R/\mathrm{pc})^3\ \ \mathrm{M}_\odot\,.$$

This gives a limiting radius of about 0.9 pc for 1 solar mass.

How does collapse occur? Rotation is removed by magnetic fields, and magnetic fields are dissipated by ambipolar diffusion over times of $10^4 - 10^5$ years, which is 10 times the dynamical time scale $1/\sqrt{G\rho}$. It is more likely that clouds are dissipated before collapse ensues, so star formation is essentially inefficient, and gas remains in the Galaxy today.

Hydrostatic equilibrium for an isothermal gas gives

$$\rho\,(r) = \frac{v_s^2}{2\pi G r^2}; \qquad m\,(r) = \frac{2v_s^2 r}{G}. \tag{4}$$

Ample observational evidence exists for this r^{-2} density dependence. “Jean’s” mass is close to previous estimate.

The cloud initially collapses isothermally because photons freely escape. M_J decreases and smaller condensed regions of the cloud become unstable to collapse: it fragments. The free-fall timescale is $\tau_{ff} = \sqrt{3/(8\pi G\bar{\rho})}$. The total energy to be radiated is of order GM^2/R, giving an estimated luminosity

$$L \simeq \frac{GM^2}{R}\sqrt{\frac{8\pi G\bar{\rho}}{3}} = \sqrt{2}G^{3/2}\left(\frac{M}{R}\right)^{5/2}.$$

This can't be larger than the blackbody luminosity $4\pi R^2\sigma T_{eff}^4$, giving

$$M \leq \left(8\pi^2 R^9\sigma^2 T_{eff}^8/G^3\right)^{1/5}.$$

Fragmentation stops when $R < R_J = 4GM_J/(9N_oT_{eff})$. Thus

$$M_J > \frac{81N_o^2}{32G}\left(\frac{9N_oT_{eff}}{2\pi^2\sigma^2G^2}\right)^{1/4} \simeq 0.005T^{1/4}\ \mathrm{M}_\odot.$$

For a temperature of 100 K, we find $M_J > 0.03\ \mathrm{M}_\odot$, greater than planetary masses, but of order of smallest stellar mass.

The Euler equations of hydrodynamics:

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial r^2\rho v}{\partial r} = 0,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{Gm(r)}{r^2},$$

where $m(r) = \int 4\pi\rho r^2 dr$ is the mass contained within the radius r. Applied to a polytropic equation of state $P = K\rho^\gamma$, these coupled partial differential equations can be replaced by ordinary differential equations. Define the self-similar variable

$$X(r,t) = K^{-1/2}G^{(\gamma-1)/2}r(\mp t)^{\gamma-2}. \qquad (5)$$

The convention is that the collapse begins at $t = -\infty$ and the central density reaches infinite values when $t = 0$. The collapse continues through positive values of t.

The hydrodynamical variables ρ, v and m can be written as dimensionless functions of X, with scale factors depending on K, G, γ and $(\mp t)$. Thus

$$\begin{aligned} \rho(r,t) &= G^{-1}(\mp t)^{-2} D(X), \\ v(r,t) &= K^{1/2} G^{(1-\gamma)/2} (\mp t)^{1-\gamma} V(X), \\ m(r,t) &= \int 4\pi\rho r^2 dr = K^{3/2} G^{(1-3\gamma)/2} (\mp t)^{4-3\gamma} M(X). \end{aligned} \tag{6}$$

Thus, a solution in terms of X tells us the behavior of a quantity *at all times at a given place* or *at all places at a given time.*

In dimensionless variables the Euler equations are:

$$\frac{D'}{D}\left[V \pm X(2-\gamma)\right] + V' = \mp 2 - 2\frac{V}{X}, \tag{7}$$

$$\gamma D^{\gamma-2} D' + V'\left[V \pm X(2-\gamma)\right] = -\frac{M}{X^2} \mp (\gamma - 1) V. \tag{8}$$

The upper (lower) sign refers to $t < 0$ $(t > 0)$.

In spherical symmetry, mass can only move radially. The equation of continuity is

$$\frac{dm(r,t)}{dt} = \frac{\partial m(r,t)}{\partial t} + v\frac{\partial m(r,t)}{\partial r} = 0.$$

This equation is easily integrated:

$$M(X) = \int 4\pi D X^2 dX = \frac{4\pi X^2 D\left[X(2-\gamma) \pm V\right]}{4 - 3\gamma}.$$

If $\gamma = 4/3$, however, we must have either $V = \mp 2X/3$ or $D = 0$.

The dimensionless Euler equations possess a singularity analogous to the one encountered in steady-state accretion and stellar winds. The physicaly acceptable solution is the one which is regular at the *critical point* where the determinant of the coupled equations Eq. (8) is zero,

$$[V(X_c) \pm X_c(2-\gamma)]^2 = \gamma D^{\gamma-1}(X_c), \tag{9}$$

and which has the correct behavior as $X \to 0$ and $X \to \infty$.

In the limit $(\mp t) \to 0$, or $r \to \infty$ for $t \neq 0$, in order to ensure that all variables are non-singular, one must have the asymptotic behavior

$$\left.\begin{aligned} D \propto & X^{-2/(2-\gamma)}; \quad V \propto X^{(1-\gamma)/(2-\gamma)} \\ & M \propto X^{(4-3\gamma)/(2-\gamma)} \end{aligned}\right\} \quad X \to \infty \tag{10}$$

For $\gamma = 4/3$, the mass becomes constant at large radii and tends to infinity for $\gamma < 4/3$. This is no problem since the solution can be terminated at large X. In the case $\gamma > 4/3$, the mass tends to zero in the large X limit which seems unphysical. For $1 < \gamma < 2$ both D and V tend to 0, and also $V \propto \sqrt{M/X}$ as $X \to \infty$, so $V \propto V_{ff} = \sqrt{2M/X}$, the free-fall velocity.

In the limit $X \to 0$, which is $r \to 0$ for $t \neq 0$, or $(-t) \to \infty$ for finite r (i.e., the initial system), one can show

$$\left.\begin{aligned} D = & D_o; \quad V = -(2/3)X \\ M = & (4\pi/3) D_o X^3 \end{aligned}\right\} X \to 0, \quad t < 0. \tag{11}$$

D_o is an integration constant which depends only on γ. The central collapsing regions are *homologous*, *i.e.*, v varies linearly with r. A boundary separates the two asymptotic behaviors for V, and the inner core from the outer core. (For $\gamma = 4/3$, there is no outer core; D falls to zero by the value $X = X_c = 2.77$.) The boundary is near the maximum velocity point, $|dV/dX| = 0$, and where the velocity equals the sound speed,

$$v = v_s = \sqrt{\partial P/\partial \rho}\,; \qquad V = A = \gamma^{1/2} D^{(\gamma-1)/2}\,. \tag{12}$$

The dimensionless sound speed is A. Homology breaks down at larger radii because information travels at a slower rate than that of the collapsing matter.

At $t = 0$ the asymptotic power-law relations Eq. (10) hold everywhere, and form initial conditions for a self-similar solution valid for $t > 0$ (lower sign of t in Eqs. (7) and (8). There is no singular point, the initial conditions determine the solution. The asymptotic relations Eq. (10) for $X \to \infty$ are still valid, but for $X \to 0$ a new set of asymptotic relations applies:

$$\left.\begin{aligned} M =& M_o; \quad V = -\sqrt{2M/X} \\ D =& \frac{\sqrt{M_o/2}}{4\pi X^{3/2}} \end{aligned}\right\} X \to 0, \quad t > 0 \tag{13}$$

The constant M_o is determined by integration. This limit corresponds, for a given r, to a sufficiently long time after the center has collapsed to infinite density. For $t < 0$, the outer core cannot go into free-fall because the inner core is collapsing too slowly. This obstacle is removed at $t = 0$, after which the outer core builds up momentum, and approaches free-fall as $X \to 0$. The free-fall collapse of the outer core is made possible by a reduction in the pressure gradient relative to gravity due to a substantial reduction in density.

The case $\gamma = 4/3$ is special in that there is no finite solution for the case $t > 0$. The solution for $t < 0$ ended at $X_c = 2.77$, with $D = 0$ for $X > X_c$. This cannot be matched to the asymptotic behavior unless $D = 0$ for $X > 0$ when $t > 0$. The hydrodynamics equation for $t < 0$ can be written as

$$\frac{4}{3} D^{-2/3} D' = -\frac{M}{X^2} \pm \frac{2X}{9}.$$

Figure 1: Self-similar solutions for $\gamma = 1.2, 1.3$ and $4/3$.

Taking a derivative after multiplying by X^2, and substituting $M' = 4\pi D X^2$, one finds

$$\frac{1}{X^2}\frac{d}{dX}\left(X^2\frac{dD^{1/3}}{dX}\right) = -\pi D + \frac{1}{6}.$$

Letting $\theta = (D/D_o)^{1/3}, \xi = X\sqrt{\pi D_o}$, we have

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3 + \frac{1}{6\pi D_o}.$$

The constant $D_o = 8.11$ from numerical integration. This is the just the Lane-Emden equation for $\gamma = 4/3$ except for the constant term. The first zero of θ now occurs at $\xi_1 = X_c\sqrt{\pi D_o} = 9.99$ as opposed to 6.854 when this term is absent. It is interesting that the value of the constant term, $(6\pi D_o)^{-1}$, is the largest value that will still produce a zero of θ; thus, when

$\xi = \xi_1$, we have $\theta_1 = \theta_1' = 0$ and $\theta_1'' = (6\pi D_o)^{-1}$. Near the origin, the density dependence is nearly identical to that of a static $\gamma = 4/3$ polytrope, but deviates from this for $X \to X_c$. Thus a collapsing $\gamma = 4/3$ polytrope has a finite extent, which is not the case for other polytropic exponents.

Table 1: Self-Similar Solutions for Collapse, $t < 0$

			V/V_{ff}							
γ	X_m	$-V_m$	m	∞	$(V/A)_\infty$	D_o	M_m	M_{ch}	$(DX^{2/(2-\gamma)})_\infty$	M_o
1.00	—	—	—	0.00	0.00	—	—	—	$1/2\pi$	0.975
1.20	4.90	1.22	0.56	0.62	2.28	0.57	11.54	8.03	0.246	
1.28	3.21	1.09	0.53	0.64	2.63	1.31	6.63	5.06	0.102	
1.29	3.06	1.09	0.54	0.65	2.76	1.50	6.28	4.90	0.083	
1.30	2.95	1.10	0.54	0.66	2.94	1.75	5.98	4.78	0.064	
1.31	2.88	1.12	0.56	0.68	3.21	2.10	5.71	4.67	0.045	
1.32	2.77	1.16	0.59	0.72	3.69	2.66	5.42	4.60	0.025	
1.33	2.77	1.28	0.66	0.80	5.11	4.00	5.07	4.56	0.006	
4/3	2.77	1.85	1.00	—	—	8.11	4.76	4.55	—	—

Now examine the behavior of the dimensionless energy

$$
\begin{aligned}
e(r,t) &= \int 4\pi\rho r^2 \left[\frac{1}{2}v^2 + \frac{K}{\gamma-1}\rho^{\gamma-1} - \frac{Gm(r)}{r}\right] dr \\
&= K^{5/2} G^{(3-5\gamma)/2} (\mp t)^{6-5\gamma} E(X)\,.
\end{aligned}
$$

As $X \to \infty$, the behavior of E satisfies $E \propto X^{(6-5\gamma)/(2-\gamma)}$, which results in finite total energy for $X < \infty$ if $2 > \gamma > 6/5$. The effective range for physical collapse solutions is then $6/5 < \gamma < 4/3$.

Isothermal Case:

However, a physical solution for the isothermal case, $\gamma = 1$, exists also for $t > 0$. We have $K = v_s^2$, so $X = v_s^{-1} r/t$. A solution can be found that has the steady state solution Eq. (4) as an initial condition for $t \to 0^+$ or $X \to \infty$,

$$D = K\left(2\pi X^2\right)^{-1}, \qquad V = 0, \qquad M = 2KX. \tag{14}$$

Near $X = 0$, Eq. (13) gives

$$\begin{aligned} m(r,t) &= \frac{v_s^3}{G} m_o t; \quad v(r,t) = \sqrt{\frac{2m_o v_s^3 t}{r}} = \sqrt{\frac{2Gm}{r}}; \\ \rho(r,t) &= \frac{1}{4\pi G}\sqrt{\frac{m_o v_s^3}{2tr^3}} = \frac{1}{4\pi}\sqrt{\frac{Gm}{tr^3}}. \end{aligned} \tag{15}$$

The parameter $m_o = 0.975$, determined by numerical integration, represents the mass that accumulates at the origin (where the density is infinite). It grows linearly with time, i.e., a constant accretion rate $\dot{m} = v_s^3 m_o/G$ exists.

In this case, a singular point is reached for $V - X = -1$, seen from Eq. (9). Since for $X > 1$, the solution is the steady state solution itself, Eq. (14), for which $V = 0$. Therefore we have that $X_c = 1$. A physical interpretation of these results is that the collapsing region begins at the center and expands linearly with the sound speed. The bottom falls out of the cloud and the collapse is inside-out. The total infalling mass, the mass within $X = 1$, is $m_{in} = 2v_s^3 t/G$, just over twice the mass accumulated at the origin, $m_{core} = m_o v_s^3 t/G$.

While the infall will be radial near the origin, at some point the centrifugal barrier becomes appreciable. With angular momentum conservation, the angular velocity $\Omega = \Omega_o (r_o/r)^2$ of each mass unit increases during infall (o refers to initial values). Ω reaches Keplerian magnitude, where centrifugal barrier balances gravity, when $r = R_C = (GM/\Omega^2)^{1/3} = \Omega_o^2 r_o^4/GM$. For 1 $M_\odot$, $\Omega_o = 2 \cdot 10^{-16}$ s^{-1}, and $r_o = 0.33$ pc, one finds $R_C \simeq 20$ AU. The formation of an extensive disc, of the order the size of planetary system, is inevitable. A protostar forms at the center, with a central density that increases in a runaway that is halted only when the core becomes optically thick.

The opacity in the collapsing cloud core, while small, nevertheless is large enough to ensure optical thickness when

$$\kappa \sim (R\rho)^{-1} \sim 5 \times 10^{-7} \left(\frac{R}{1\mathrm{AU}} \right)^2 \mathrm{cm}^2 \ \mathrm{g}^{-1}. \tag{16}$$

For a core mass $M = m_o v_s^3/G$, this occurs for $R \simeq 1$ AU if $T \simeq 15$ K. Beyond this point, the collapse will be nearly adiabatic since the surface temperature is so small. The abrupt halt of the central collapse unleashes a shock, which runs out through the infalling matter, raising its temperature and ionizing it.

With the Virial Theorem, $n = 3/2$, and assuming complete ionization (kinetic energy equals ionization energy),

$$K = -\frac{1}{2}\Omega = \frac{3}{7}\frac{GM^2}{R} = I = MN_o \left(\chi_H X + \chi_{He} Y \right), \tag{17}$$

where the ionization potentials $\chi_H (\chi_{He}) = 15.8(19.8)$ eV. For a solar composition, $I/MN_o \simeq 17$ eV. Inverting, we find $R \simeq 50\ \mathrm{R}_\odot = 0.23$ AU, close to the above value. From the relation $K \simeq 3MN_o\bar{T}/(2\mu)$, with $\mu = 0.6$, we find $\bar{T} \sim 9 \times 10^4$ K.

Consider slow contraction of a convective protostar. For $T_p < 5000$ K, where ionization is incomplete and $s \simeq -10$, T_p is extremely insensitive to L and M. With the correct constants

$$
\begin{aligned}
L &\simeq 0.024 \left(\frac{M}{\mathrm{M}_\odot}\right)^{4/7} \left(\frac{R}{\mathrm{R}_\odot}\right)^{16/7} \ \mathrm{L}_\odot \\
&\simeq 1.64 \cdot 10^4 \left(\frac{\mathrm{M}_\odot}{M}\right)^{4} \left(\frac{T_p}{3500\ \mathrm{K}}\right)^{32} \ \mathrm{L}_\odot .
\end{aligned}
\tag{18}
$$

Therefore, in an Hertzsprung-Russell diagram, a collapsing protostar moves vertically along a line of nearly fixed temperature, the so-called *Hayashi track*. The luminosity $L \propto M^{4/7} R^{16/7}$ decreases with contraction. For fixed M, an $n = 3/2$ polytrope has $R \propto \sqrt{K} \propto e^{s/3}$: the star contracts only if the entropy per particle falls. The contraction timescale is easily found from

$$
\tau_{cont} \simeq -\frac{E}{L} \simeq 5.3 \times 10^8 \left(\frac{M}{M_\odot}\right)^{10/7} \left(\frac{R}{R_\odot}\right)^{-23/7} \ \mathrm{yr}, \tag{19}
$$

which increases rapidly as the star shrinks. It also decreases rapidly with mass, since for a given value of I/M, $R \propto M$: thus $\tau_{cont} \propto M^{-13/7}$. At the beginning of the Hayashi track, when $R \sim 50 R_\odot$ for a $1 M_\odot$ protostar, $\tau_{cont} \simeq 1400$ yr, but much greater than the free fall time.

For very massive protostars, accretion occurs all the way to the main sequence. For less massive stars, accretion ceases during protostar collapse. How does a star know when to stop accretion? Essentially, this happens when the central temperature rises to the value needed to ignite deuterium, about 10^6 K. Thus, using the perfect gas law with the $n = 3/2$ polytrope relations, we find the "stellar birthline" $R/R_\odot \simeq 8.3 M/\mathrm{M}_\odot$ and $L \simeq 3(M/\mathrm{M}_\odot)^{20/7}\ \mathrm{L}_\odot$. Deuterium ignition drives stellar winds that effectively halt further accretion.

The protostar leaves the Hayashi track when L falls below the minimum for a fully convective star. With $\mu = 0.6$

$$
L_{min} = \frac{5.8}{\eta_c} \left(\frac{M}{M_\odot}\right)^{5.5} \left(\frac{R_\odot}{R}\right)^{.5} L_\odot . \tag{20}
$$

A fully convective star does not quite have a uniform energy generation, but one that is proportional to the temperature. For a perfect gas law,

$T = P\mu/N_o\rho \propto \rho^{1/n} \propto \theta$, where θ is the Lane-Emden variable. Therefore

$$\eta_c = \frac{T_c M}{\int T dM} = \frac{\xi_1^2 \theta_1'}{\int \theta d\left(\xi^2\theta'\right)} = -\frac{\xi_1^2\theta_1'}{\int \xi^2\theta^{n+1} d\xi} \tag{21}$$

which is approximately 2 for all polytropes. (Direct integration gives 5/2, 2 and $16/3\pi$ for the cases n=0, 1, and 5, respectively). Using $\eta_c \simeq 2.5$ and equating Eqs. (18) and (20),

$$L_{min} \simeq 1.0 \left(\frac{M}{\mathrm{M}_\odot}\right)^{60/13} \mathrm{L}_\odot; \quad R_{min} \simeq 5.1 \left(\frac{M}{M_\odot}\right)^{23/13} \mathrm{R}_\odot. \tag{22}$$

Although this luminosity is about 50% too large, the minimum radius too large by an even greater factor, compared to more detailed calculations, probably owing to poor assumptions about the opacity. Also

$$T_{c,min} = \frac{G\mu M}{N_o\left(n+1\right)R_{min}\left(-\xi_1\theta_1'\right)} \simeq 1.4 \cdot 10^6 \left(\frac{M_\odot}{M}\right)^{10/13} \mathrm{K}, \tag{23}$$

which is comfortably above the temperature where deuterium burning occurs. From Eq. (19), the time needed to collapse from infinity to a given radius is $(7/23)\tau_{cont}$, so the time needed to collapse to R_{min} is

$$t_{min} \simeq 7.6 \times 10^5 \left(\frac{M_\odot}{M}\right)^{399/91} \mathrm{yr}. \tag{24}$$

From the Hayashi Track to the Main Sequence:

During collapse, the Virial Theorem says that half the energy goes into radiation and the other half into heat (if n fixed):

$$L = -\frac{1}{2}\frac{dE}{dt} = \frac{n-3}{5-n}\frac{GM^2}{2R^2}\frac{dR}{dt};$$
$$\frac{dL}{dt} = -\frac{1}{2}\frac{d^2E}{dt^2} = \frac{n-3}{5-n}\frac{GM^2}{2R^2}\left[-\frac{2}{R}\left(\frac{dR}{dt}\right)^2 + \frac{d^2R}{dt^2}\right].$$

Constancy of T_{eff} with $L = 4\pi\sigma R^2 T_{eff}^4$ implies $dL/L = 2dR/R$ and $d^2R/dt^2 = 4(dR/dt)^2/R$. This leads to a solution

$$R = R_o\left(1 + \frac{6L_oR_ot}{GM^2}\right)^{-1/3},$$

where R_o and L_o refer to the initial radius and luminosity.

Continued collapse to the main sequence involves a change in the polytropic structure from $n = 3/2$ to $n = 3$ (the standard model). If we neglect this, however, and use the hydrostatic condition $d^2I/dt^2 = 0$ where $I \propto MR^2$ is the moment of inertia, one finds that $Rd^2R/dt^2 = -(dR/dt)^2$. This implies that $dL/L = -3dR/R$, which is now positive, and to a solution

$$R = R_o\sqrt{1 - \frac{4R_oL_ot}{GM^2}}\,.$$

This also implies that $d\ln T_{eff}/d\ln R = -5/4$ and $d\ln L/d\ln T_{eff} = 12/5$. This phase of collapse is known as the *Henyey* phase, and results in an increase in L and T_{eff} with stellar shrinkage. The stellar radius, for a one solar mass star, must shrink from R_{min} to $\mathrm{R}_\odot$, the effective temperature rises from about 3500 K to 5500 K, so $L_{min} \simeq (3500/5500)^{12/5} \simeq 3\ \mathrm{L}_\odot$ and $R_{min} \simeq (5500/3500)^{4/5} \simeq 0.7\ \ \mathrm{R}_\odot$. The slope in the H-R diagram of the Henyey track is 2.4, smaller than that of the main sequence, so the Henyey trajectory will eventually intersect it.

Note from the above that L_{min} varies with mass less steeply than does L_{ssm}. Since we expect that for solar mass stars that $L_{min}/L_{ssm} < 1$ (although we didn't get this result in the above), in principle as mass is decreased there will be a point at which $L_{min} = L_{ssm}$. For small enough masses, the Henyey phase essentially disappears, and a lower limit to stellar masses occurs.

Main Sequence Structure

The main sequence is defined as those stars that burn hydrogen to helium in their cores. Lower mass stars do this via the pp cycle, and higher mass stars via the CNO cycle. The latter is much more temperature sensitive, and leads to convective cores in massive stars. The adiabatic temperature gradient is

$$\left.\frac{dT}{dr}\right|_{ad} = -\frac{2}{5}\frac{T}{P}\frac{dP}{dr} = -\frac{8\pi}{15}\frac{\mu G\rho_o r}{N_o} \tag{25}$$

for a perfect gas near the center. Diffusive transport leads to

$$\left.\frac{dT}{dr}\right|_{rad} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2} \simeq -\frac{\kappa_o\rho_c^2\epsilon_c r}{acT_c^3}, \tag{26}$$

where we used the Thomsen opacity (valid at high temperatures) and approximated $L(r) \simeq (4\pi/3)\rho_c\epsilon_c r^3$. The condition for convective instability is simply $|dT/dr|_{rad} > |dT/dr|_{ad}$, or

$$\epsilon_c > \frac{8\pi}{15}\frac{\mu GacT_c^3}{N_o\kappa_o\rho_c} \simeq 1.2\times 10^4 \left(\frac{T_{c,6}}{14}\right)^3 \left(\frac{150\ \mathrm{g\ cm^{-3}}}{\rho_c}\right) \mathrm{erg\ g^{-1}s^{-1}}. \quad (27)$$

For comparison, the pp cycle has an energy generation rate of

$$\epsilon_{pp} \simeq 17 \left(\frac{T_6}{14}\right)^4 \left(\frac{\rho}{150\ \mathrm{g\ cm^{-3}}}\right) \mathrm{erg\ g^{-1}s^{-1}}, \quad (28)$$

so the Sun's core must be radiative. In a massive star, for CNO, we have

$$\epsilon_{CNO} \simeq 4\times 10^5 \left(\frac{T_6}{25}\right)^{17} \left(\frac{\rho}{150\ \mathrm{g\ cm^{-3}}}\right) \mathrm{erg\ g^{-1}s^{-1}}; \quad (29)$$

these stars have convective cores.

If energy generation is very temperature sensitive, approximate the core as having a point-like energy source. Thus, for $r > 0$, we have $L(r) = L$. We look for a power-law solution for T outside the convective core: $T = br^m$. With the radiative transport equation,

$$m = -\frac{1}{4} \qquad b^4 = \frac{3\kappa_o\rho_c L}{16\pi ac} \quad (30)$$

if ρ is constant. The convective core's radius will be determined by $|dT/dr|_{rad} = |dT/dr|_{ad}$:

$$r_{core} = \left(\frac{3\kappa_o L}{16\pi ac}\right)^{1/9} \left(\frac{15N_o}{32\pi\mu G}\right)^{4/9} \rho_c^{-1/3}. \quad (31)$$

Using the relation $\rho_c \propto M/R^3$ we have

$$r_{core}/R \propto L^{1/9}/M^{1/3} \propto M^{5/18} \quad (32)$$

using the standard solar model for $L(M)$. Convective cores increase with stellar mass.

Red Giant Structure

There are two important characteristics for the red giant structure. First is that the core develops into an isothermal structure, supported largely by electron degeneracy pressure. Second is that the envelope evolves to very low density and hence a very large radius.

Isothermal Core

Towards the end of the main sequence life, the stellar core is largely composed of He, with a pronounced gradient of mean molecular weight. As the core mass increases, and H is exhausted in the stellar center, the active burning layer moves outward. The core becomes increasingly degenerate, and since degenerate matter is highly conductive, the temperature gradient in the core declines. The core temperature is then established by the temperature in the active burning layer, of order 10^7 K.

Rarified Envelope

Although some of the large expansion of the outer layers can be attributed to the abrupt change in the mean molecular weight and entropy at the core/envelope interface which produce a density discontinuity there (see discussion conerning the u, v variables), probably the most important reason is due to the large energy production within the hydrogen-burning shell exterior to the isothermal helium core. Effectively, no energy is produced in the envelope, so $L(r)$ can be treated as constant there. We posit power law solutions to the diffusion and hydrostatic equilibrium equations in the envelope of the form

$$P = P_s \left(\frac{r}{R_s}\right)^{-a} \qquad T = T_s \left(\frac{r}{R_s}\right)^{-b} \tag{33a}$$

$$\rho = \rho_s \left(\frac{r}{R_s}\right)^{-c} \qquad M = M_s \left(\frac{r}{R_s}\right)^{d}. \tag{33b}$$

With Kramer's opacity, $a = 42/11, b = 10/11, c = 32/11, d = 1/11$, so the polytropic index is $n = c/b = 3.2$. We find

$$\rho_s = \alpha \frac{3M_s}{4\pi R_s^3} = \frac{1}{44\pi}\frac{M_s}{R_s^3} \qquad T_s = \frac{11}{42}\frac{\mu G M_s}{N_o R_s} \tag{34}$$

so the value of α from above is $1/33$. Utilizing the diffusion equation to solve for the luminosity, we find

$$L = \frac{16\pi ac}{3\kappa_o}\frac{10}{11}\frac{R_s T_s^{7.5}}{\rho_s^2} = 2490 \left(\frac{M_s}{M_\odot}\right)^{11/2} \left(\frac{R_\odot}{R_s}\right)^{1/2} L_\odot. \tag{35}$$

With $T_s \simeq 10^7$ K, $M_s \simeq M_\odot$ we find $R_s \simeq 0.4R_\odot$. Substituting into Eq. (33a) using $T = T_p \sim 3500$ K, we find $R = 2500\mathrm{R}_\odot = 2.3$ AU, very large indeed. The total mass, using Eq. (33b), is $M = (10^7/3500)^{1/10} \simeq 2.2\mathrm{M}_\odot$, so the envelope mass is comparable to the core mass. Large L and R are inevitable.

Discontinuities in the mean molecular weight and entropy exist also. Because of P and T continuity, an abrupt decrease in μ is accompanied by a corresponding decrease in ρ and $|dP/dr|$. An abrupt increase in s also results in a decrease in ρ. Both result in expansions of the outer layers and L in a kind of runaway. The core of a star with a molecular weight or entropy discontinuity is denser than the core of a star withut one, since ρ must increase.

Expansion factors of red giant envelopes are 100-500 times, but μ gradients or core shrinkage produce only 50–100% changes. The remainder can be traced to isothermality of the core.

Assume T, M and R obey power law relations just outside the core and well into the envelope as in Eq. (33). Then

$$T_s = \frac{\mu G M_s}{N_o\,(n+1)\,R_s}. \tag{36}$$

Since T_s is fixed by nuclear requirements during shell burning, this suggests that $R_s \propto M_s$. However, M_s increases as the shell slowly burns outward in mass, and the core's center contracts due to the higher gravity. Nevertheless, R_s slowly increases. The quantity

$$u = \frac{d\ln M}{d\ln r} = \frac{3\rho\,(r)}{\bar{\rho}} = \frac{4\pi r^3 \rho\,(r)}{M\,(r)} \tag{37}$$

is very small just beyond the shell. Expanding $\rho(r)$ and $M(r)$ about the origin in an isothermal core,

$$\rho = \rho_c\left(1 - \frac{2\pi G\rho_c\mu}{3N_oT_c}R_s^2\right) \qquad M = \frac{4\pi}{3}\rho_c R_s^3\left(1 - \frac{2\pi\rho_c\mu}{5N_oT_c}R_s^2\right) \tag{38}$$

we find

$$u = 3\left(1 - \frac{4\pi\mu G\rho_c}{15N_oT_c}R_s^2\right) \simeq 3\left[1 - 0.07\rho_c\frac{3\times10^7\mathrm{K}}{\mathrm{T_c}}\left(\frac{R_s}{R_\odot}\right)^2\right] \tag{39}$$

which rapidly decreases as ρ_c increases and R_s increases. For comparison, for an $n = 3$ polytrope, the factor of 0.07 will be reduced a factor of 4 for the same conditions. The radius of the star swells as a result, since $\ln R \propto \int u^{-1} d \ln M$.

Chandrasekhar-Schönberg Mass Limit

An isothermal core develops in stars whose mass is less than about 6 $M_\odot$. For larger stars, the core mass fraction, when hydrogen shell burning begins, exceeds the so-called *Chandrasekhar-Schönberg limit*

$$M_{cs} = 0.37 \left(\frac{\mu_e}{\mu_c}\right)^2 M, \tag{40}$$

for which an isothermal core is unable to support the envelope's weight. Here μ_e and μ_c are the mean molecular weight of the envelope and core, respectively. These stars do not expand as dramatically as less massive stars.

The relation Eq. (40) can be motivated from a dimensional analysis of the core and the virial theorem. Apply the virial theorem to the core alone:

$$\frac{3N_oT_cM_c}{n\mu_c} + \frac{3}{5-n}\frac{GM_c}{R_c} = 4\pi R_c^3 P_c \tag{41}$$

The term on the right-hand side stems from the non-zero pressure P_c at the edge of the isothermal core (with temperature T_c, mass M_c, molecular weight μ_c and radius is R_c). Solving for P_c yields:

$$P_c = \frac{3}{4\pi R_c^3}\left[\frac{N_oT_cM_c}{n\mu_c} - \frac{GM_c^2}{5R_c}\right].$$

The maximum core radius that can be supported is found by maximizing P_c with respect to R_c, which gives

$$R_{c,max} = \frac{4n\mu_c GM_c}{15N_oT_c}; \qquad P_{c,max} = \frac{3^4}{4^5\pi M_c^2}\left(\frac{5}{G}\right)^3\left(\frac{N_oT_c}{n\mu_c}\right)^4.$$

Since $T_c \propto \mu M/R$, and $P_c \propto M^2/R^4$, where M and R are for the whole star, one finds that

$$q_c = \frac{M_c}{M} \propto \left(\frac{\mu}{\mu_c}\right)^2.$$

An accurate derivation yields a proportionality factor 0.37. Thus, an isothermal core supports at most 37% of the total mass, but if the core is mostly He, this is reduced to 10%.

Stars less than about 15 $M_{\odot}$ develop degenerate cores, which leads to the well-known *Helium flash* behavior at the tip of the red giant branch. Partially degenerate cores don't obey the Chandrasekhar-Schönberg limit and support larger masses. As red giants expand, and T_p cools, they approach the Hayashi track and are nearly completely convective.