Compact Stars – White Dwarfs, Planets, Neutron Stars

White dwarf structure dominated by $P(\rho)$ for degenerate electron gas.

$$R = \left[\frac{(n+1)\,K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/(2n)} \xi_1$$

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)\,K}{4\pi G}\right]^{n/(n-1)} \xi_1^{(3-n)/(n-1)} \left(-\xi_1^2 \theta_1'\right)$$

$$= 4\pi \left[\frac{(n+1)\,K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/(2n)} \left(-\xi_1^2 \theta_1'\right).$$

Non-relativistic $\gamma = 5/3$:

$$K = \frac{\left(3\pi^2\right)^{2/3}}{5} \frac{\hbar^2}{m_e} \left(N_o Y_e\right)^{5/3} = 10^{13} Y_e^{5/3} \ \text{ cgs}$$

$$R = 1.121 \times 10^4 \rho_{c,6}^{-1/6} \left(2Y_e\right)^{5/6} \ \text{ km}$$

$$M = 0.496 \rho_{c,6}^{1/2} \left(2Y_e\right)^{5/2} \ \text{ M}_\odot = 0.701 \left(\frac{R}{10^4 \ \text{km}}\right)^{-3} \left(2Y_e\right)^5 \ \text{ M}_\odot. \tag{1}$$

Relativistic $\gamma = 4/3$:

$$K = \frac{\left(3\pi^2\right)^{1/3}}{4} \hbar c \left(N_o Y_e\right)^{4/3} = 1.24 \times 10^{15} Y_e^{4/3} \ \text{ cgs}$$

$$R = 3.35 \times 10^4 \rho_{c,6}^{-1/3} \left(2Y_e\right)^{2/3} \ \text{ km} \tag{2}$$

$$M = 1.457 \left(2Y_e\right)^2 \ \text{ M}_\odot.$$

Very low density (Thomas-Fermi regime $\gamma = 10/3$):

$$K = \frac{3^{1/3}\pi^3 e^2}{10}\left(\frac{4\pi N_o}{A}\right)^{10/3}\left(\frac{3\hbar c}{m_e c^2}\frac{\hbar c}{e^2}\right)^6 = 1.05\times 10^{13}\left(\frac{12}{A}\right)^{10/3} \quad \text{cgs}$$
$$R = 1.18\times 10^5\left(\frac{12}{A}\right)^{5/3}\rho_c^{2/3} \ \text{km}$$
$$M = 0.001915\left(\frac{12}{A}\right)^5 \rho_c^3 \ \mathrm{M}_\odot$$
$$= 2.88\times 10^{-8}\left(\frac{A}{12}\right)^{5/2}\left(\frac{R}{10^4 \ \text{km}}\right)^{9/2} \ \mathrm{M}_\odot. \tag{3}$$

Physical reasoning behind the Chandrasekhar mass:

Consider N degenerate fermions in a star of radius R, so that number density $n \propto NR^{-3}$. Momentum of a fermion is $\sim \hbar n^{1/3}$ and Fermi energy is $E_F \sim \hbar c N^{1/3} R^{-1}$. The gravitational energy per fermion is $\sim -GMm_B R^{-1}$ if $M = Nm_B$. The total energy is

$$E = E_F + E_G = \hbar c N^{1/3}/R - GNm_b^2/R.$$

Equilibrium is reached when this is minimized. Both terms scale as $1/R$.

When E is positive, E can be decreased by increasing R. This decreases E_F so that eventually the fermions become non-relativistic: then $E_F \sim p_F^2 \sim R^{-2}$. This then decreases faster than E_G, so E becomes negative. However, as $R \to \infty$, $E \to 0$. This implies there is a minimum of E at a finite value of R.

When E is negative, E can be decreased without bound by decreasing R so that no equilibrium state is possible and a black hole forms.

The maximum baryon number for equilibrium is determined by setting $E = 0$:

$$N_{max} \sim \left(\frac{\hbar c}{G m_b^2}\right)^{3/2} \sim 2 \times 10^{57}$$
$$M_{max} \sim N_{max} m_B \sim 1.5 \ \mathrm{M}_\odot .$$

Note that the mass is independent of the fermion's mass.

The radius at equilibrium is set by the condition $E_F \geq mc^2$,

$$R \leq \frac{\hbar c}{mc^2}\left(\frac{\hbar c}{G m_B^2}\right)^{1/2} \sim \begin{cases} 5 \times 10^3 \ \mathrm{km}, & m = m_e \\ 3 \ \mathrm{km}, & m = m_n \,. \end{cases}$$

At sufficiently high density, neutronization and pyconuclear reactions can occur. Thus, both A and $N-Z$ will increase with density. The neutronization threshold for ^{56}Fe is about 10^9 g cm^{-3}. At this density, the Fermi energy of an electron is about $m_c c^2$+3.695 MeV, the threshold for the inverse beta-decay ^{56}Fe + $e^- \rightarrow ^{56}$Mn + ν_e. The Mn immediately electron captures: ^{56}Mn + $e^- \rightarrow ^{56}$Cr + ν_e. The Cr is stable until densities above 10^{10} g cm^{-3} are reached.

Lighter nuclei have other thresholds: ^{4}He is at 20.6 MeV, ^{12}C is at 13.4 MeV, ^{16}O is at 10.4 MeV and ^{20}Ne is at 7.0 MeV. The loss of electrons softens the EOS: the Chandrasekhar mass decreases. A white dwarf at these densities will begin to gravitationally collapse. Thus the maximum density $\lesssim 10^{10}$ g cm^{-3}, with a minimum radius $\gtrsim$ 1500 km.

Electrostatic corrections

In a degenerate system in which the nuclei are ordered in either a solid or liquid, there is Coulomb energy associated with the ordering. If the nuclei are equally spaced and surrounded by a uniform density electron gas, the interaction energy per electron is

$$E_c/Z = -\frac{9}{10}\frac{Ze^2}{R_c} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3}e^2 n_e^{1/3}$$

where $n_e = 3Z/(4\pi R_c^3)$. The corresponding pressure is

$$P_c = n_e^2 \frac{\partial E_c/Z}{\partial n_e} = -\frac{3}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3}e^2 n_e^{4/3}.$$

In the extreme relativistic limit this is just a constant fraction (a few percent) of the degeneracy pressure:

$$\frac{P_c}{P_d} = -\frac{2^{5/3}}{5}\left(\frac{3}{\pi}\right)^{1/3}\frac{e^2}{\hbar c}Z^{2/3}.$$

In the non-relativistic limit, P_c becomes more important at lower and lower densities:

$$\frac{P_c}{P_d} = -\frac{m_e e^2 Z^{2/3}}{(2n_e)^{1/3}\,\pi\hbar^2}.$$

At low enough density, $P_c = -P_d$:

$$n_{e,c} = \frac{Z^2\left(m_e e^2\right)^3}{2\pi^3\hbar^6} \qquad \left(\rho_c \simeq 0.4Z^2 \text{ g cm}^{-3}\right),$$

and the total pressure vanishes. For iron this is about 250 g cm^{-3}, which is not the laboratory value of 7.86 g cm^{-3} because it is incorrect to treat the e^- gas as uniform.

The electron Fermi energy, modified by Coulomb potential

$$E_F = -eV(r) + \frac{p_F^2}{2m_e}.$$

is constant in space, otherwise electrons would move to a lower E_F. The electron density is

$$n_e = \frac{8\pi}{3h^3} p_F^3 = \frac{8\pi}{3h^3}\left[2m_e\left(E_F + eV(r)\right)\right]^{3/2}.$$

The potential is determined by Poisson's equation

$$\nabla^2 V = 4\pi e n_e + \text{ nuclear contribution}$$

where the last term is effectively a delta function at the origin. Omitting it for $r > 0$, we have the boundary condition: $rV(r) \to Ze$ as $r \to 0$. The electric field should vanish at the outer boundary, since this volume must be overall neutral: $dV/dr|_{R_c} = 0$. Poisson's equation in spherical geometry is

$$\frac{d^2\phi}{dx^2} = \frac{\phi^{3/2}}{x^{1/2}}, \tag{4}$$

$$E_F + eV(r) = \frac{Ze^2\phi(r)}{r}, \quad x = r\left(\frac{128Z}{9\pi^2}\right)^{1/3}\frac{m_e e^2}{\hbar^2}.$$

Eq. (4) has boundary conditions

$$\phi(0) = 1, \qquad \phi(x_o)' \equiv \frac{d\phi}{dx}\Big|_{x_o} = \frac{\phi(x_o)}{x_o},$$

where x_o corresponds to $r = R_c$. The latter condition can be seen by evaluation of $Z = 4\pi\int n_e r^2 dr$ over the entire volume. The equation Eq. (4) has a unique solution, when $\phi'(0) = -1.588071$, in that as $x_o \to \infty$, $\phi(x_o) \to 144x_o^{-3} \to 0$. Otherwise, for larger values of $\phi'(x_o)$, ϕ doesn't vanish anywhere and diverges as $x \to \infty$. At some point, the second boundary condition will be satisfied.

The pressure at the outer boundary is that of free particles

$$P = \frac{8\pi}{15h^3 m_e} p_F^5 \left(R_c\right) = \frac{1}{10\pi} Z^2 e^2 \left(\frac{128Z}{9\pi^2}\right)^{4/3} \left(\frac{m_e e^2}{\hbar^2}\right)^4 \left[\frac{\phi\left(x_o\right)}{x_o}\right]^{5/2} .$$

The density is the total mass divided by the volume:

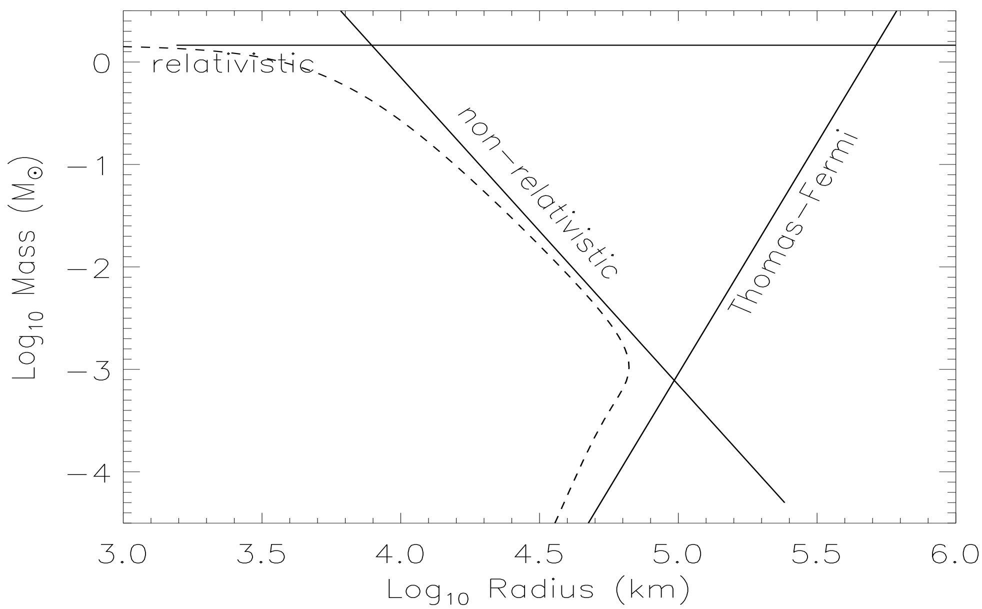
$$\rho = \frac{3Am_B}{4\pi} \frac{128Z}{9\pi^2} \left(\frac{m_e e^2}{\hbar^2 x_o}\right)^3 = \frac{4Am_B Z}{3} \left(\frac{2m_e e^2}{\pi\hbar^2 x_o}\right)^3 .$$

For low densities, the solution approaches the unique solution $\phi(x_o) \to 144 x_o^{-3}$. Thus, $P \propto \rho^{10/3}$, with K given by Eq. (3).

Mass-Radius Relation for Degenerate Objects

The mass-radius diagram for cold compact objects is shown in the figure: the solid lines are the limiting expressions Eqs (1–3), the dashed line is the full result, for ^{12}C. The maximum radius configuration has the properties, approximately, of the planet Jupiter.

In the relativistic limit, for radii much smaller than 5000 km, the equation of state will deviate from that of a $\gamma = 4/3$ gas. Electron capture will reduce Y_e and the value of the Chandrasekhar mass. Therefore, a regime where $dM/d\rho_c < 0$ will exist. Such a regime is dynamically unstable. At sufficiently high density, where nuclear forces become important, the effective value of γ will increase, the mass will reach a minimum value ($M_{min} \simeq 0.01$ M$_\odot$, where $R \simeq 300$ km), and stability is restored. As the central density increases further, $dM/d\rho_c > 0$. This is the neutron star regime. As the mass increases, and the radius shrinks, general relativity, which we have heretofore ignored, becomes important. The most important feature that general relativity introduces is that at densities well in excess of the nuclear saturation density, $\rho_s = 2.7 \cdot 10^{14}$ g cm^{-3}, the mass reaches a maximum value, in the range 1.5-3 M$_\odot$. Larger density configurations are once again dynamically unstable. The maximum mass is discussed in a subsequent lecture.



Cooling of white dwarfs

The interior of a white dwarf has energy transport dominated by conduction. The electrons are extremely degenerate, nowever, so they must have very large mean free paths. The thermal conductivity is very high. The temperature gradient must be rather small. The interior is roughly isothermal. Near the surface, isothermality breaks down as the opacity increases. The surface regions are diffusive, with a temperature gradient

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}.$$

At the high densities, Kramer's opacity is dominant: $\kappa = \kappa_o \rho T^{-3.5}$, with $\kappa_o \simeq 4.3 \times 10^{24} Z(1+X)$ cm^2 g^{-1}. With hydrostatic equilibrium,

$$\frac{dP}{dT} = \frac{4ac}{3}\frac{4\pi G m\,(r)}{\kappa_o L}\frac{T^{6/5}}{\rho}.$$

The surface layer is thin, so $m(r) = M$. Using the nondegenerate pressure $P = N_o \rho kT/\mu$, and eliminating ρ, we have

$$PdP = \frac{4ac}{3}\frac{4\pi GM}{\kappa_o L}\frac{kN_o}{\mu}T^{7.5}dT.$$

Integrating from $P = 0$ at $T = 0$ to the interior,

$$\rho = \sqrt{\frac{2}{8.5}\frac{4ac}{3}\frac{4\pi GM}{\kappa_o L}\frac{\mu}{kN_o}}T^{3.25}.$$

The surface approximation breaks down in the interior when matter becomes degenerate. This occurs when the non-degenerate pressure equals the degenerate pressure at radius r_* where one has ρ_* and T_*. This results in

$$\rho_* = 2/4 \times 10^{-8} T_*^{3/2} Y_e^{-1} \text{ g cm}^{-3},$$

$$L = 5.7 \times 10^5 \frac{\mu Y_e^2}{Z\,(1+X)}\frac{M}{\mathrm{M}_\odot}T_*^{3.5} \text{ ergs}^{-1}.$$

This is similar to the blackbody law $L = 4\pi R^2 \sigma T_{eff}^4$, but involves the interior temperature, not the visible temperature of the surface. It suggests a relation like $T_{eff} \propto T_*^{7/8}$. From L and M, and the composition, one can deduce T_*. L of $10^{-2} - 10^{-5}$ $\mathrm{L}_\odot$ imply $T_* = 10^6 - 10^7$ K, $\rho_* < 10^3$ g cm^{-3} and

$$\frac{R - r_*}{R} \simeq 4.25 \frac{RN_o kT_*}{GM\mu} \lesssim 10^{-2}.$$

The energy that is radiated as thermal energy by the white dwarf is the residual ion thermal energy, since the electrons are degenerate and the star can't release any gravitational energy. For a monatomic non-degenerate ion gas, with $c_v = 3k/2$ the total thermal energy is (with $T_* = T$)

$$U = \frac{3}{2}\frac{N_o k}{A}TM.$$

The cooling rate is $L = -dU/dt$. Using $L = CMT^{7/2}$,

$$t = t_o + \frac{3}{5}\frac{kN_o}{AC}\left(T^{-5/2} - T_o^{-5/2}\right).$$

Taking $T << T_o$, the cooling time is

$$\tau = \frac{3}{5}\frac{N_o kT}{A}\frac{M}{L} = \frac{3}{5}\frac{N_o k}{CA}\left(\frac{CM}{L}\right)^{5/7}.$$

For $L \sim 10^{-3}$ $L_{\odot}$, we obtain $\tau \sim 10^9$ yr.

It is interesting to compare the cooling theory with observations. Like cars on a highway, the slower they go, the more congested the freeway (or vice versa). The number of white dwarfs of a given luminosity should relate to their relative abundance, especially if the birth rate of white dwarfs has been roughly constant in time. The luminosity function is $\phi(L)$ which is the space density of white dwarfs per unit interval of $\log L$. Thus, with a uniform production rate,

$$\phi\left(L\right) \propto \left[\frac{d\log\left(L\right)}{dt}\right]^{-1}.$$

If $\tau \propto L^{-\alpha}$, where our theory suggests $\alpha \simeq 5/7$, one finds

$$\log\phi = -\alpha\log L + \text{ constant.}$$

It turns out this is approximately matched by observations, until $L \lesssim 10^{-4}$ $L_{\odot}$. Theoretical corrections to the specific heat of very cold white dwarfs imply that $\alpha \to 0$ below this luminosity, but observations actually reveal that $\alpha << 0$ when $L \lesssim 10^{-4.5}$. This deficit of white dwarfs of low luminosity is due to the finite age of the galactic disc. The cooling time of the white dwarfs where the sudden drop in α occurs then yields an estimate of the age of the galactic disc, about 10 billion years.