Convective Energy Transport

Schwarzschild criterion for convective instability:

Consider a rising bubble. Pressure equilibrium maintained if $v < v_{sound}$. If no energy exchange with surroundings (bubble is adiabatic). Bubble exands and cools. Upward motion will continue if $T_{bubble} > T$ at new position, since perfect gas law $P = N_o \rho T/\mu$ implies $\rho_{bubble} < \rho$ at new position (μ unchanged). If locally in radiative equilibrium, this occurs if

$$\left.\frac{dT}{dr}\right|_{ad} < \left.\frac{dT}{dr}\right|_{rad}$$

since gradients are negative. Convective instability occurs if

$$\nabla_{ad} = \left.\frac{d\ln T}{d\ln P}\right|_{ad} < \left.\frac{d\ln T}{d\ln P}\right|_{rad} = \nabla_{rad}.$$

Adiabatic temperature gradient:

$$dQ = dE + PdV, \quad C_V = \left.\frac{\partial E}{\partial T}\right|_V, \quad C_P = \left.\frac{\partial E}{\partial T}\right|_P.$$

With $PV = N_o T$ and $V = \mu/\rho$, find $C_P = C_V + N_o$ ($k_B = 1$). Define $\gamma = C_P/C_V$. For adiabatic change $dQ = 0$

$$\left.\frac{d\ln T}{d\ln P}\right|_{ad} = \nabla_{ad} = \frac{\gamma - 1}{\gamma}.$$

For monatomic gas, $C_V = (3/2)N_o, C_P = (5/2)N_o, \gamma = 5/3, \nabla_{ad} = 2/5$.

Convection can occur if either C_V becomes large (*e.g.*, ionization) or $|dT/dr|_{rad}$ becomes large (*e.g.*, intense nuclear energy generation).

Instability condition rewritten:

$$\left.\frac{dT}{dr}\right|_{rad} = -\frac{3\kappa_R \rho}{4acT^3}\frac{L(r)}{4\pi r^2} > \left(1 - \gamma^{-1}\right)\frac{T}{P}\frac{dP}{dr},$$

$$L(r) \geq \frac{16\pi acG}{3\kappa_R}\left(1 - \gamma^{-1}\right)\frac{T^4}{P} m(r).$$

For ideal nondegenerate gas,

$$\eta(r) \geq 0.62 \frac{\mu T_6^3}{\kappa_R \rho}\frac{M}{\mathrm{M}_\odot}\frac{\mathrm{L}_\odot}{L}.$$

This is, for the ssm using electron scattering for the opacity at the center, $\eta_c \geq 30.6(\frac{M}{\mathrm{M}_\odot})^3(\frac{\mathrm{L}_\odot}{L})$. For the Sun, η_c is much larger than the value for H burning, 9.8. For lower masses, electron scattering is not a good approximation. For higher mass stars, η_c is smaller, while η for CNO burning is 40. Thus, stars of greater than $1.5\mathrm{M}_\odot$ have convective cores.

With convection, energy flux carried by both radiation and convection:

$$L(r) = L_{rad} + L_{conv}, \quad L_{rad} = -4\pi r^2 \frac{4ac}{3\kappa_R \rho} T^3 \frac{dT}{dr}.$$

Convecting matter has upward and downward mass flows (g/cm^2/s): $\rho_{u,d} v_{u,d}$. Heat contents per g: $e_{u,d} = c_P T_{u,d}$. For no net mass flow, and $\rho_u = \rho_d = \rho$, one has $v_u = v_d = \bar{v}$. Net energy transport is

$$L_{conv} = 4\pi r^2 \rho \bar{v} c_P (T_u - T_d) = 4\pi r^2 \rho \bar{v} c_P \Delta T.$$

Mixing Length Theory:

$$\Delta T = -\left(\frac{dT}{dr} - \left.\frac{dT}{dr}\right|_{ad} \right) \ell = \ell \Delta\nabla T, \quad \Delta\rho = -\left(\frac{d\rho}{dr} - \left.\frac{d\rho}{dr}\right|_{ad} \right) \ell = \ell \Delta\nabla\rho.$$

For perfect monatomic nondegenerate gas, since $\Delta\nabla P = 0$,

$$\Delta\nabla\rho = \frac{\rho}{T}\left(1 - \frac{d\ln\mu}{d\ln T}\right)\Delta\nabla T = Q\Delta\nabla T.$$

Velocity determined by buoyancy force per unit volume: $F = g\Delta\rho$ with $g = Gm/r^2$ the local gravity. Net acceleration per gram is F/ρ.

$$\frac{v^2}{2} = \int_0^\ell \frac{F}{\rho} d\ell = gQ \frac{\Delta\nabla T}{T} \frac{\ell^2}{2}.$$

$$\bar{v} = \frac{v}{2} = \frac{\ell}{2}\sqrt{\frac{gQ\Delta\nabla T}{T}}.$$

$$L_{conv} = 4\pi r^2 \rho c_P \sqrt{\frac{gQ}{T}} (\Delta\nabla T)^{3/2} \frac{\ell^2}{2}.$$

Choice of ℓ:

$$\ell \approx \frac{dr}{d\ln P} = \frac{N_o T}{g\mu} \simeq \frac{T(r)}{\mathrm{T}_\odot} \frac{\mathrm{M}_\odot}{m(r)} \left(\frac{r}{\mathrm{R}_\odot}\right)^2 \mathrm{R}_\odot.$$

In interior, $\ell \approx \mathrm{R}_\odot/10$. The constraint that $L_{conv} < L$ implies

$$\Delta\nabla T < \left(\frac{4Gr}{15\ell^2 N_o k_B}\right)^{2/3} \left(\frac{T}{g}\right)^{1/3}.$$

For the interior of the ssm, this is

$$\Delta\nabla T < 4 \cdot 10^{-16} \text{ K cm}^{-1} << |dT/dr|;$$

when convection occurs, the temperature gradient is effectively adiabatic.