General Relativity and Compact Objects – Neutron Stars and Black Holes

We confine attention to spherically symmetric configurations. The metric for the static case can generally be written

$$ds^2 = e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - e^{\nu(r)}dt^2. \tag{1}$$

Einstein's equations for this metric are:

$$\begin{aligned} 8\pi\rho(r) &= \frac{1}{r^2}\left(1 - e^{-\lambda}\right) + e^{-\lambda}\frac{\lambda'(r)}{r}, \\ 8\pi p(r) &= -\frac{1}{r^2}\left(1 - e^{-\lambda}\right) + e^{-\lambda}\frac{\nu'(r)}{r}, \\ p'(r) &= -\frac{p(r) + \rho(r)}{2}\nu'(r). \end{aligned} \tag{2}$$

Derivatives with respect to the radius are denoted by $'$. We employ units in which $G = c = 1$, so that 1 $M_\odot$ is equivalent to 1.475 km. The first of Eq. (2) can be exactly integrated. Defining the constant of integration so obtained as $m(r)$, the enclosed gravitational mass, one finds

$$e^{-\lambda} = 1 - 2m(r)/r, \qquad m(r) = 4\pi\int_0^r \rho r'^2 dr'. \tag{3}$$

The second and third of Einstein's equations form the equation of hydrostatic equilibrium, also known as the Tolman-Oppenheimer-Volkov (TOV) equation in GR:

$$\frac{-p'(r)}{\rho(r) + p(r)} = \frac{\nu'(r)}{2} = \frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}. \qquad r \le R \tag{4}$$

Near the origin, one has $\rho'(r) = p'(r) = m(r) = 0$. Outside the distribution of mass, which terminates at the radius R, there is vacuum with $p(r) = \rho(r) = 0$, and Einstein's equations give

$$m(r) = m(R) = M, \quad e^{\nu} = e^{-\lambda} = 1 - \frac{2M}{r}, \qquad r \ge R \tag{5}$$

the Schwarzschild solution. The black hole limit is seen to be $R = 2M$, which is 2.95 km for 1 $M_\odot$.

From thermodynamics, if there is uniform entropy per nucleon, the first law gives

$$0 = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right)$$

where n is the number density. If e is the internal energy per nucleon, we have $\rho = n(m + e)$. From the above, $p = n^2 de/dn$, so that

$$d\left(\log n\right) = \frac{d\rho}{\rho + p} = -\frac{1}{2}\frac{d\rho}{dP}d\nu, \qquad dn = \frac{d\rho}{h},$$

where $h = (\rho + p)/n$ is the enthalpy per nucleon or the chemical potential. The constant of integration for the number density can be established from conditions at the surface of the star, where the pressure vanishes (it is not necessary that the energy density or the number density also vanish there). If $n = n_o, \rho = \rho_o$ and $e = e_o$ when $P = 0$, one finds $\rho_o - mn_o = n_o e_o$ and

$$mn\left(r\right) = \left(\rho\left(r\right) + p\left(r\right)\right) e^{(\nu(r) - \nu(R))/2} - n_o e_o. \tag{6}$$

Another quantity of interest is the total number of nucleons in the star, N. This is not just M/m (m being the nucleon mass) since in GR the binding energy represents a decrease of the gravitational mass. The nucleon number is

$$N = \int_0^R 4\pi r^2 e^{\lambda/2} n\left(r\right) dr = \int_0^R 4\pi r^2 n\left(r\right)\left[1 - \frac{2m\left(r\right)}{r}\right]^{-1/2} dr, \tag{7}$$

and the total binding energy is

$$BE = Nm - M. \tag{8}$$

Analytic Solutions to Einstein's Equations

It turns out there are many analytic solutions to Einstein's equations. However, there are only 3 that satisfy the criteria that the pressure and energy density vanish on the boundary R, and that the pressure and energy density decrease monotonically with increasing radius. Many others, in fact an infinite number, are known that have vanishing pressure, but not energy density, at R.

Among the simplest analytic solutions is the so-called Schwarzschild interior solution for an incompressible fluid, $\rho(r) =$ constant. In this case,

$$\begin{aligned} m\left(r\right) &= \frac{4\pi}{3}\rho r^3, \qquad e^{-\lambda} = 1 - 2\beta\left(r/R\right)^2, \\ e^{\nu} &= \left[\frac{3}{2}\sqrt{1-2\beta} - \frac{1}{2}\sqrt{1-2\beta\left(r/R\right)^2}\right]^2, \\ p\left(r\right) &= \frac{3\beta}{4\pi R^2}\frac{\sqrt{1-2\beta} - \sqrt{1-2\beta\left(r/R\right)^2}}{\sqrt{1-2\beta\left(r/R\right)^2} - 3\sqrt{1-2\beta}}, \\ \rho &= n\left(m+e\right) = \text{ constant}, \qquad n = \text{ constant}. \end{aligned} \tag{9}$$

Here, $\beta \equiv M/R$. Clearly, $\beta < 4/9$ or else the central pressure will become infinite. It can be shown that this limit to β holds for *any* star. This solution is technically unphysical for the reasons that the energy density does not vanish on the surface, and that the speed of sound, $c_s = \sqrt{\partial p/\partial \rho}$ is infinite. The binding energy for the incompressible fluid is analytic (taking $e = 0$):

$$\frac{BE}{M} = \frac{3}{4\beta}\left(\frac{\sin^{-1}\sqrt{2\beta}}{\sqrt{2\beta}} - \sqrt{1-2\beta}\right) - 1 \simeq \frac{3\beta}{5} + \frac{9\beta^2}{14} + \cdots \tag{10}$$

In the case that e/m is finite, the expansion becomes

$$\frac{BE}{M} \simeq \left(1 + \frac{e}{m}\right)^{-1}\left[-\frac{e}{m} + \frac{3\beta}{5} + \frac{9\beta^2}{14} + \cdots\right]. \tag{11}$$

In 1967, Buchdahl discovered an extension of the Newtonian $n = 1$ polytrope into GR that has an analytic solution. He assumed an equation of state

$$\rho = 12\sqrt{p_* p} - 5p \tag{12}$$

and found

$$\begin{aligned} e^{\nu} &= \left(1-2\beta\right)\left(1-\beta-u\right)\left(1-\beta+u\right)^{-1}; \\ e^{\lambda} &= \left(1-2\beta\right)\left(1-\beta+u\right)\left(1-\beta-u\right)^{-1}\left(1-\beta+\beta\cos Ar'\right)^{-2}; \\ 8\pi p &= A^2u^2\left(1-2\beta\right)\left(1-\beta+u\right)^{-2}; \\ 8\pi\rho &= 2A^2u\left(1-2\beta\right)\left(1-\beta-3u/2\right)\left(1-\beta+u\right)^{-2}; \\ mn &= 12\sqrt{pp_*}\left(1-\frac{1}{3}\sqrt{\frac{p}{p_*}}\right)^{3/2}; \qquad c_s^2 = \left(6\sqrt{\frac{p_*}{p}} - 5\right)^{-1}. \end{aligned} \tag{13}$$

Here, p_* is a parameter, and r' is, with u, a radial-like variable

$$
\begin{aligned}
u &= \beta\left(Ar'\right)^{-1}\sin Ar'; \\
r' &= r\left(1-\beta+u\right)^{-1}\left(1-2\beta\right); \\
A^2 &= 288\pi p_*\left(1-2\beta\right)^{-1}.
\end{aligned}
\tag{14}
$$

For this solution, the radius, central pressure, energy and number densities, and binding energy are

$$
\begin{aligned}
R &= (1-\beta)\sqrt{\frac{\pi}{288p_*\left(1-2\beta\right)}}; \\
p_c &= 36p_*\beta^2; \quad \rho_c = 72p_*\beta\left(1-5\beta/2\right); \quad n_c m_n c^2 = 72\beta p_*\left(1-2\beta\right)^{3/2}; \\
\frac{BE}{M} &= \left(1-1.5\beta\right)\left(1-2\beta\right)^{-1/2}\left(1-\beta\right)^{-1} - 1 \approx \frac{\beta}{2} + \frac{\beta^2}{2} + \frac{3\beta^3}{4} + \cdots.
\end{aligned}
\tag{15}
$$

This solution is limited to values of $\beta < 1/6$ for $c_{s,c} < 1$.

In 1939, Tolman discovered that the simple density function $\rho = \rho_c[1-(r/R)^2]$ has an analytic solution. It is known as the Tolman 7 solution:

$$
\begin{aligned}
&e^{-\lambda} = 1-\beta x\left(5-3x\right), \qquad e^{\nu} = \left(1-5\beta/3\right)\cos^2\phi, \\
&P = \frac{1}{4\pi R^2}\left[\sqrt{3\beta e^{-\lambda}}\tan\phi - \frac{\beta}{2}\left(5-3x\right)\right], \qquad n = \frac{\left(\rho+P\right)}{m}\frac{\cos\phi}{\cos\phi_1}, \\
&\phi = \left(w_1-w\right)/2+\phi_1, \qquad \phi_c = \phi\left(x=0\right), \\
&\phi_1 = \phi\left(x=1\right) = \tan^{-1}\sqrt{\beta/\left[3\left(1-2\beta\right)\right]}, \\
&w = \log\left[x-5/6+\sqrt{e^{-\lambda}/\left(3\beta\right)}\right], \qquad w_1 = w\left(x=1\right).
\end{aligned}
\tag{16}
$$

In the above, $x=(r/R)^2$. The central values of P/ρ and the square of the sound speed $c_{s,c}^2$ are

$$
\left.\frac{P}{\rho}\right|_c = \frac{2\tan\phi_c}{15}\sqrt{\frac{3}{\beta}} - \frac{1}{3}, \qquad c_{s,c}^2 = \tan\phi_c\left(\frac{1}{5}\tan\phi_c + \sqrt{\frac{\beta}{3}}\right). \tag{17}
$$

There is no analytic result for the binding energy, but in expansion

$$
\frac{BE}{M} \approx \frac{11\beta}{21} + \frac{7187\beta^2}{18018} + \frac{68371\beta^3}{306306} + \cdots. \tag{18}
$$

This solution is limited to $\phi_c < \pi/2$, or $\beta < 0.3862$, or else P_c becomes infinite. For causality $c_{s,c} < 1$ if $\beta < 0.2698$.

In 1950, Nariai discovered yet another analytic solution. It is known as the Nariai 4 solution, and is expressed in terms of a parametric variable r':

$$e^{-\lambda} = \left(1 - \sqrt{3\beta}\left(\frac{r'}{R'}\right)^2 \tan f\left(r'\right)\right)^2, \quad e^{\nu} = (1-2\beta)\frac{e^2}{c^2}\left(\frac{\cos g\left(r'\right)}{\cos f\left(r'\right)}\right)^2,$$

$$f\left(r'\right) = \cos^{-1} e + \sqrt{\frac{3\beta}{4}}\left[1 - \left(\frac{r'}{R'}\right)^2\right], \quad g\left(r'\right) = \cos^{-1} c + \sqrt{\frac{3\beta}{2}}\left[1 - \left(\frac{r'}{R'}\right)^2\right],$$

$$r = \frac{e}{c}\frac{r'}{\cos f\left(r'\right)}\sqrt{1-2\beta},$$

$$p\left(r'\right) = \frac{\cos f\left(r'\right)}{4\pi R'^2}\frac{c^2}{e^2}\sqrt{3\beta}\Big[\sqrt{2}\cos f\left(r'\right)\tan g\left(r'\right)$$
$$\left[1 - \sqrt{3\beta}\left(\frac{r'}{R'}\right)^2 \tan f\left(r'\right)\right] - \sin f\left(r'\right)\left[2 - \frac{3}{2}\sqrt{3\beta}\left(\frac{r'}{R'}\right)^2 \tan f\left(r'\right)\right]\Big],$$

$$\rho\left(r'\right) = \frac{\sqrt{3\beta}}{4\pi R'^2\sqrt{1-2\beta}}\frac{c^2}{e^2}$$
$$\left[3\sin f\left(r'\right)\cos f\left(r'\right) - \sqrt{\frac{3\beta}{4}}\left(\frac{r'}{R'}\right)^2\left(3 - \cos^2 f\left(r'\right)\right)\right],$$

$$m\left(r'\right) = \frac{r'^3}{R'^2}\frac{e}{c}\frac{\tan f\left(r'\right)}{\cos f\left(r'\right)}\sqrt{3\beta\left(1-2\beta\right)}\left[1 - \sqrt{\frac{3\beta}{4}}\left(\frac{r'}{R'}\right)^2 \tan f\left(r'\right)\right]. \tag{19}$$

The quantities e and c are

$$e^2 = \cos^2 f\left(R'\right) = \frac{2+\beta+2\sqrt{1-2\beta}}{4+\beta/3}$$
$$c^2 = \cos^2 g\left(R'\right) = \frac{2e^2}{2e^2 + \left(1-e^2\right)\left(7e^2-3\right)\left(5e^2-3\right)^{-1}}.$$

The pressure-density ratio and sound speed at the center are

$$\frac{P_c}{\rho_c} = \frac{1}{3}\left(\sqrt{2}\cot f\left(0\right)\tan g\left(0\right) - 2\right),$$
$$c_{s,c}^2 = \frac{1}{3}\left(2\tan^2 g\left(0\right) - \tan^2 f\left(0\right)\right).$$

The central pressure and sound speed become infinite when $\cos g(0) = 0$ or when $\beta = 0.4126$, and the causality limit is $\beta = 0.223$. This solution is quite similar to Tolman 7.

Neutron Star Maximum Mass

The TOV equation can be scaled by introducing dimensionless variables:

$$p = q\rho_o, \quad \rho = d\rho_o, \quad m = z/\sqrt{\rho_o}, \quad r = x/\sqrt{\rho_o},$$

$$\frac{dq}{dx} = -\frac{(q+d)\left(z + 4\pi d x^3\right)}{x\,(x-2z)}, \qquad \frac{dz}{dx} = 4\pi d x^2 dx. \tag{20}$$

Rhoades and Ruffini showed that the causally limiting equation of state

$$p = p_o + \rho - \rho_o \qquad\qquad \rho > \rho_o \tag{21}$$

results in a neutron star maximum mass that is practically independent of the equation of state for $\rho < \rho_o$, and is

$$M_{max} = 4.2\sqrt{\rho_s/\rho_o}\ \mathrm{M}_\odot. \tag{22}$$

Here $\rho_s = 2.7 \cdot 10^{14}$ g cm^{-3} is the nuclear saturation density. One also finds for this equation of state that

$$R_{max} = 18.5\sqrt{\rho_s/\rho_o}\ \mathrm{km}, \qquad \beta_{max} \simeq 0.33\,. \tag{23}$$

Since the most compact configuration is achieved at the maximum mass, this represents the limiting value of β for causality.

Some analytic motivation for the above results was given by Nauenberg and Chapline. They assumed that in the interior of a star both n and ρ were constant, so P is also because of the first law. The TOV equation is not satisfied for this assumption, however, so the results of this analysis are very approximate. The baryon number for fixed n and ρ is

$$\begin{aligned} N &= 4\pi \int_0^R \frac{nr^2 dr}{\sqrt{1 - 8\pi\rho r^2/3}} = 4\pi n \left(\frac{3}{8\pi\rho}\right)^{3/2} \int_0^\chi \sin^2\theta d\theta \\ &= 2\pi n \left(\frac{3}{8\pi\rho}\right)^{3/2} (\chi - \sin\chi\cos\chi)\,, \end{aligned} \tag{24}$$

where $\sin\theta = \sqrt{2m(r)/r}$ and $\sin\chi = \sqrt{2\beta}$. In terms of χ, we can write the gravitational mass as

$$M = \sqrt{\frac{3}{32\pi\rho}} \sin^3\chi.$$

As χ increases, n, ρ and p in the star increase, and the mass M reaches a maximum for $\chi < \pi/2$. To guarantee stability, the total nucleon number N must also be maximized, which is equivalent to the equation

$$\left.\frac{\partial M}{\partial \chi}\right|_N = 0.$$

This results in a pair of equations:

$$\frac{d\rho}{\rho} = 6\frac{\cos\chi}{\sin\chi}d\chi, \qquad 4\sin^2\chi d\chi = (\chi - \sin\chi\cos\chi)\left(3\frac{d\rho}{\rho} - 2\frac{dn}{n}\right).$$

Combining this with the first law $d\rho/dn = (\rho + p)/n$, we obtain

$$\frac{p}{\rho} = \frac{6\cos\chi\,(\chi - \sin\chi\cos\chi)}{9\chi\cos\chi - 9\sin + 7\sin^3\chi}.$$

The condition that $p/\rho < \infty$ limits $\sin\chi < 0.985$, and $p/\rho < 1$ limits $\sin\chi < 0.956$. The further condition $dp/d\rho < 1$ limits $\sin\chi < 0.90$, which is equivalent to $\beta < 0.405$. Note this value is significantly larger than the limit obtained above, because of the less restrictive conditions. Nevertheless, we can now derive a maximum mass by employing the maximal equation of state Eq. (21). Rewriting this equation as

$$\rho > \frac{\rho_o - P_o}{1 - P/\rho},$$

and applying it to the mean density of the star $\rho = 3M/(4\pi R^3)$, using $\beta = M/R$, we find

$$M = \sqrt{\frac{3\beta^3}{4\pi\rho}} < \sqrt{\frac{3\beta^3}{4\pi\rho_o}(1 - p/\rho)}.$$

It is valid to have taken $p_o << \rho_o$. In geometrized units, the nuclear saturation density $\rho_s = 2.7 \cdot 10^{14}$ g cm^{-3} has the equivalence $\rho_s^{-1/2} = 70$ km or 45.5 $\mathrm{M}_\odot$. Therefore,

$$M < 45.5\sqrt{\frac{3\beta^3}{4\pi}\frac{\rho_s}{\rho_o}(1 - p/\rho)}\ \mathrm{M}_\odot.$$

With $\beta = 0.405$ and $p/\rho = 0.364$, the limiting mass is $M < 4.57\sqrt{\rho_s/\rho_o}$ $\mathrm{M}_\odot$.

For the Buchdahl solution at the causal limit, $\beta = 1/6$ and $p/\rho = \beta/(2-5\beta)$, which lead to

$$M = (1-\beta)\sqrt{\frac{\pi\beta^3\,(1-5\beta/2)}{4\,(1-2\beta)\,\rho_c}} < 2.14\sqrt{\rho_s/\rho_c}\ \mathrm{M}_\odot.$$

For the Tolman 7 solution at the causal limit, $\beta \simeq 0.27$ and $p/\rho = 2/(\sqrt{75\beta}) \simeq 0.44$, which lead to

$$M = \sqrt{\frac{15\beta^3}{8\pi\rho_c}} < 4.9\sqrt{\rho_s/\rho_c}\ \mathrm{M}_\odot.$$

Finally, for the Nariai 4 solution at the causal limit, $\beta \simeq 0.228$ and $p/\rho \simeq 0.246$, which lead to

$$M = \frac{\beta}{\cos f\,(R')}\sqrt{\frac{3^{3/2}\beta^{1/2}\sin f\,(0)\cos f\,(0)}{4\pi\rho_c}} < 3.4\sqrt{\rho_s/\rho_c}\ \mathrm{M}_\odot.$$

Maximal Rotation Rates for Neutron Stars

The absolute maximum rotation rate is set by the "mass-shedding" limit, when the rotational velocity at the equatorial radius (R) equals the Keplerian orbital velocity $\Omega = \sqrt{GM/R^3}$, or

$$P_{min} = 0.55\left(\frac{10\ \mathrm{km}}{R}\right)^{3/2}\left(\frac{M}{\mathrm{M}_\odot}\right)^{1/2}\ \mathrm{ms}. \tag{25}$$

However, the actual limit on the period is larger because rotation induces an increase in the equatorial radius. In the so-called Roche model, one treats the rotating star as being highly centrally compressed. For an $n = 3$ polytrope, $\rho_c/\bar{\rho} \simeq 54$, so this would be a good approximation. In more realistic models, such as $\rho = \rho_c[1-(r/R)^2]$, for which $\rho_c/\bar{\rho} = 5/2$, and an $n = 1$ polytrope, for which $\rho_c/\bar{\rho} = \pi^2/3$, this approximation is not as good. Using it anyway, the gravitationl potential near the surface is $\Phi_G = -GM/r$ and the centrifugal potential is $\Phi_c = -(1/2)\Omega^2 r^2\sin^2\theta$, and the equation of hydrostatic equilibrium is

$$(1/\rho)\,\nabla P = \nabla h = -\nabla\Phi_G - \nabla\Phi_c, \tag{26}$$

where $h = \int dP/\rho$ is the enthalpy per unit mass. Integrating this from the surface to an interior point along the equator, one finds

$$h\,(r) - GM/r - (1/2)\,\Omega^2 r^2 = K = -GM/r_e - (1/2)\,\Omega^2 r_e^2,$$

where r_e is the equatorial radius and $h(r_e) = 0$. We assume $K = -GM/R$, the value obtained for a non-rotating configuration. The potential $\Phi \equiv \Phi_G + \Phi_c$ is maximized at the point where $\partial\Phi/\partial r|_{r_c} = 0$, or where $r_c^3 = gM/\Omega^3$ and $\Phi = -(3/2)GM/r_c$. Thus, r_e has the largest possible value when $r_e = r_c = 3R/2$, or

$$\Omega^2 = \frac{GM}{r_c^3} = \left(\frac{2}{3}\right)^3 \frac{GM}{R^3}. \tag{27}$$

The revised minimum period then becomes

$$P_{min} = 1.0 \left(\frac{10\ \text{km}}{R}\right)^{3/2} \left(\frac{M}{\text{M}_\odot}\right)^{1/2} \quad \text{ms}. \tag{28}$$

Calculations including general relativity show that the minimum spin period for an equation of state can be accurately expressed in terms of its maximum mass and the radius at that maximum mass as:

$$P_{min} \simeq 0.82 \left(\frac{10\ \text{km}}{R_{max}}\right)^{3/2} \left(\frac{M_{max}}{\text{M}_\odot}\right)^{1/2} \quad \text{ms}. \tag{29}$$

It is interesting to compare the rotational kinetic energy $T = I\Omega^2/2$ with the gravitational potential energy W at the mass-shedding limit. I is the moment of inertia about the rotation axis:

$$I = \frac{8\pi}{3}\int_0^R r^4 \rho dr$$

for Newtonian stars. (In GR, one must take into account frame-dragging as well as volume and redshift corrections.) Using $\Omega^2 = (2/3)^3 GM/R^3$, we can write $T = \alpha(2/3)^3 GM^2/R$ and $|W| = \beta GM^2/R$. We have $\alpha = 1/5, \beta = 3/5$ for an incompressible fluid; $\alpha = 1/3 - 2/\pi^2, \beta = 3/4$ for an $n = 1$ polytrope; $\alpha = 0.0377, \beta = 3/2$ for an $n = 3$ polytrope; $\alpha = 1/7, \beta = 5/7$ for $\rho = \rho_c[(1-(r/R)^2]$. We therefore find that $T/|W|$ is 0.0988, 0.0516, 0.00745 and 0.0593, respectively, for these four cases, at the mass-shedding limit. For comparison, an incompressible ellipsoid becomes secularly (dynamically) unstable at $T/|W| = 0.1375(0.2738)$, much larger values.