Binary Stars

Consider a binary composed of two stars of masses M_1 and M_2. We define $M = M_1 + M_2$ and $\mu = M_1M_2/M$. If a_1 and a_2 are the mean distances of the stars from the center of mass, then $M_1a_1 = M_2a_2$. The mean separation of the stars is $a = a_1 + a_2$. If the orbit is elliptical with eccentricity e, then the separation at periastron is $a(1-e)$ and at apastron it is $a(1+e)$. The total energy and angular momentum of the binary are

$$E = -\frac{1}{2}\frac{GM_1M_2}{a} \qquad J = \mu\sqrt{GaM\left(1-e^2\right)} = \mu\Omega a^2\sqrt{1-e^2}.$$

Kepler's Law is

$$\Omega^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{GM}{a^3}.$$

The projected orbital velocity of star 1 is

$$v_1 = \Omega a_1 \sin i.$$

The quantity

$$f_1\left(M_1, M_2, i\right) = \frac{\left(M_2 \sin i\right)^3}{M^2} = \frac{v_1^3}{G\Omega}$$

is known as the mass function since it depends only on observables v_1, P. If Doppler shifts from star 2 are measured, then

$$f_2\left(M_1, M_2, i\right) = \frac{\left(M_1 \sin i\right)^3}{M^2} = \frac{v_2^3}{G\Omega}$$

can also be found. Then

$$\frac{M_2}{M_1} = \frac{v_1}{v_2}$$

independent of i. If the binary is eclipsing, the angle i can be determined and the masses individually determined as well.

Mass Transfer and Roche Lobes

The total potential of a binary is

$$-\Phi = \frac{GM_1}{r_1} + \frac{GM_2}{r_2} + \frac{1}{2}d^2\Omega^2,$$

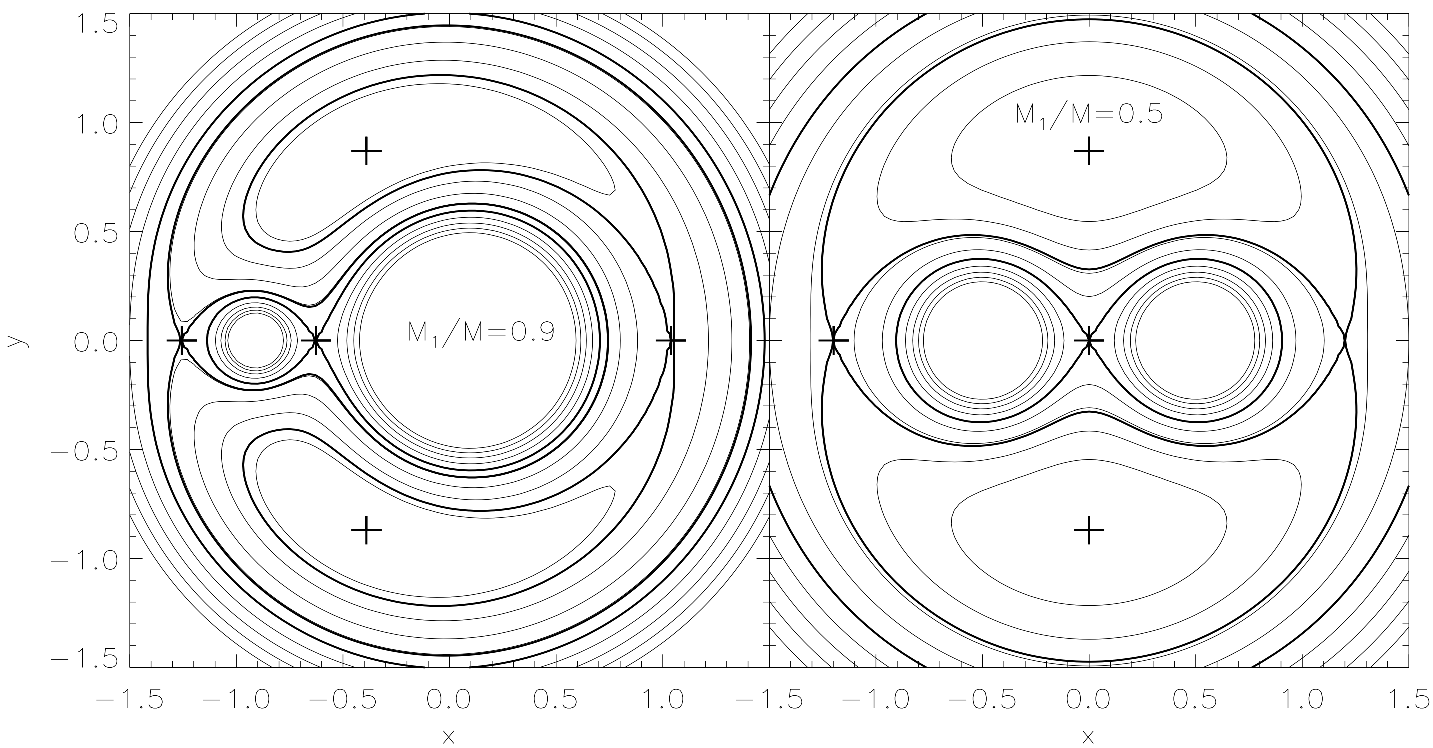
where r_1 and r_2 are the distances to stars 1 and 2 and d is the distance to the rotation axis. Restricting ourselves to the orbital plane, with the origin at the center of mass,

$$-\Phi\left(x, y\right) = \frac{GM_1}{\sqrt{\left(x-a_1\right)^2 + y^2}} + \frac{GM_2}{\sqrt{\left(x+a_2\right)^2 + y^2}} + \frac{1}{2}\left(x^2 + y^2\right)\frac{GM}{a^3}.$$

In dimensionless coordinates $\bar{x} = x/a$, $\bar{y} = y/a$, $m_1 = M_1/M$, $m_2 = M_2/M$:

$$-\Phi\left(\bar{x}, \bar{y}\right) = \frac{GM}{a}\left[\frac{m_1}{\sqrt{\left(\bar{x} - m_2\right)^2 + \bar{y}^2}} + \frac{m_2}{\sqrt{\left(\bar{x} + m_1\right)^2 + \bar{y}^2}} + \frac{1}{2}\left(\bar{x}^2 + \bar{y}^2\right)\right].$$

Contours of constant Φ are shown in the figure. There are deep minima at the stellar centers, and maxima at five so-called Lagrangian points. The L_1 point between the stars is significant because if a star expands and reaches the potential surface passing through it, mass can be transferred to its companion.



The equipotential surface that passes through L_1 is called the Roche lobe, and its size depends upon the mass ratio of the binary. Kopal (1959) gives for the radius R_R with nearly the same volume as the Roche lobe:

$$\frac{R_R}{a} = 0.46\left(\frac{M_1}{M}\right)^{1/3}. \qquad (1)$$

A better fit is by Eggleton:

$$R_R/a = 0.49\left[.6 + \left(\frac{M_1}{M_2}\right)^{-2/3}\ln\left(1 + \left(\frac{M_1}{M_2}\right)^{1/3}\right)\right]^{-1}. \qquad (2)$$

Assume the binary is circular. Then

$$a = \frac{MJ^2}{GM_1^2 (M - M_1)^2}, \qquad da = \left(\frac{2a}{M_1}\right)\left(\frac{2M_1 - M}{M - M_1}\right) dM_1, \tag{3}$$

if $dM = dJ = 0$. This shows that if $M_2 < M_1$, transferring mass from M_1 to M_2 results in a shrinkage of the orbit. Am episode of *conservative* mass transfer in a binary results in

$$a_{final} = a_{initial} \left(\frac{M_{1,initial} M_{2,initial}}{M_{1,final} M_{2,final}}\right)^2 .$$

Eq. (3) implies that in terms of the mass ratio of the binary, $q = M_2/M_1$,

$$da = \frac{2a}{q}\left(\frac{q-1}{1+q}\right) dq, \tag{4}$$

or $a \propto (1+q)^4 q^{-2}$. Expressing Eq. (1) in terms of q, then taking the derivative and combining with Eq. (4),

$$dR_L = R_L \frac{da}{a} - \frac{R_L}{3}\frac{dq}{1+q} = R_L \left[\frac{2}{q}\left(\frac{q-1}{1+q}\right) - \frac{1}{3(1+q)}\right] dq.$$

This implies that the Roche lobe size reaches its minimum value when $q = 6/5$, or $M_1 = 5M/11$.

On the other hand, suppose that mass is lost from one star in the form of a wind and is not accreted onto the companion. Then we might expect that

$$a_{final} = a_{initial} \frac{M}{M - \Delta M},$$

and mass loss will cause an increase in a binary's separation.

Now consider mass transfer when star 1 fills its Roche lobe. Stable mass transfer occurs when the change in radius of star 1 after transferring an increment of mass through the inner Lagrangian point is not offset by a corresponding change in the Roche radius, triggered by the new mass ratio of the binary. This requires that the logarithmic change of radius with mass for star 1 satisfies

$$\frac{d\ln R}{d\ln M_1} \equiv \alpha \geq \frac{d\ln R_R}{d\ln M_1} = \frac{d\ln a}{d\ln M_1} + \frac{1}{3} = 2\left(\frac{2M_1 - M}{M - M_1}\right) + \frac{1}{3}.$$

In an equal mass binary, the first term vanishes. Generally, we can expect that this condition is generally satisfied. It is not, however, for a star with a convective envelope, for which $\gamma = 5/3$ and $R \propto M^{-1/3}$.

In some situations, mass transfer will be driven by losses of orbital angular momentum. The primary sources of angular momentum loss are magnetic braking and gravitational radiation. We have

$$\frac{\dot{a}}{a} = 2\frac{\dot{J}}{J} - 2\left(1 - \frac{M_2}{M_1}\right)\frac{\dot{M}_2}{M_2},$$

where the donor star is taken to be 2, so that $\dot{M}_2 < 0$. Using the simple Roche lobe formula,

$$\frac{\dot{R}_R}{R_R} = 2\frac{\dot{J}}{J} - 2\left(1 - \frac{M_2}{M_1}\right)\frac{\dot{M}_2}{M_2} + \frac{1}{3}\frac{\dot{M}_2}{M_2}.$$

Assume that $\dot{R}_2/R_2 = \alpha(\dot{M}_2/M_2$, where $\alpha = -1/3$ for a non-relativistic degenerate, or convective, star, and $\alpha = 1$ for a main sequence star. For stable mass transfer, R_2 should remain equal to R_R. Then we have

$$\frac{\dot{J}}{J} = \left(\frac{5}{6} + \frac{\alpha}{2} - \frac{M_2}{M_1}\right)\frac{\dot{M}_2}{M_2}.$$

Since both sides of this equation must be negative, we find

$$\frac{M_2}{M_1} \le \frac{5}{6} + \frac{\alpha}{2}.$$

When $\alpha = -1/3(1)$, $M_2/M_1 \le 2/3(4/3)$. Gravitational radiation leads to

$$\frac{\dot{J}}{J} = -\frac{32G^3}{5c^5}\frac{M_1 M_2 (M_1 + M_2)}{a^4} \ \mathrm{s}^{-1}. \tag{5}$$

Explosive Mass Loss

Another case of mass transfer occurs after a supernova explosion, but here the mass loss is sudden and catastrophic and the companion does not accept the mass. If too much mass is lost from the system, the binary will be disrupted. The survival of the binary depends on the amount of mass loss, the phase of the binary (i.e., is it near periastron (a more stable situation) or apastron (a less stable situation)), and the magnitude and direction of any “kick” imparted to the supernova remnant star. If the explosion is perfectly symmetric, the kick velocity is zero. However, if the explosion is asymmetric, the kick velocity is determined by momentum conservation. A 1% asymmetry in the neutrinos released in a supernova could impart a kick velocity of approximately

$$\Delta V = \frac{\Delta E}{M_{ns}c} \simeq 360 \ \ \mathrm{kms}^{-1}$$

assuming a neutron star mass of $M_{ns} = 1.4$ M$_\odot$ and $\Delta E = 3 \times 10^{51}$ erg. This, in many cases, exceeds the relative orbital velocities of the stars prior to the explosion. As we will see, a kick in the direction of orbital motion of the remnant destabilizes the system, but a reverse kick can stabilize it.

Following J. Hills, *Ap. J.* **267** (1983) 322, the total energy of the binary prior to the explosion is

$$E_o = -\frac{GM_1^oM_2}{2a_o} = -\frac{GM_1^oM_2}{r} + \frac{1}{2}\mu_o V_o^2$$

where the subscript o refers to the initial system. r is the instantaneous separation of the two stars (equal to a_o if the orbit is circular). At periastron, $r = a_o(1-e_o)$ and at apastron, $r = a_o(1+e_o)$. $\mu_o = M_1^oM_2/M_o$ is the reduced mass and M_o is the initial total mass. V_o is the initial relative velocities of the two stars. We can furthermore define a parameter $V_c^2 = GM_o/a_o$ as the relative velocity in the circular orbit case.

Immediately after the explosion, if the orbit is still bound, we have

$$E = -\frac{GM_1M_2}{2a} = -\frac{GM_1M_2}{r} + \frac{1}{2}\mu V^2$$

by comparison, where $\mu = M_1M_2/M$. Defining $\Delta M = M_1^o - M_1 = M_o - M$, it is straightforward to demonstrate that

$$\frac{a}{a_o} = \frac{1 - \Delta M/M_o}{1 - (2a_o/r)\left(\Delta M/M_o\right) + \left(V_o^2 - V^2\right)/V_c^2}.$$

In order to remain bound, it is therefore necessary that $a < \infty$, or

$$\frac{\Delta M}{M_o} < \frac{r}{2a_o}\left(1 - \frac{V_o^2 - V^2}{V_c^2}\right).$$

Several cases can now be considered. If the explosion occurs on a timescale short compared to the orbital period, and if no kick is imparted to the supernova remnant star, it is reasonable to assume that $V = V_o$, i.e., the relative velocities of the two stars remains the same. In this case, the condition for stability becomes

$$\frac{\Delta M}{M_o} < \frac{r}{2a_o}.$$

If the orbit is initially circular, the RHS of the above is simply 1/2. If the initial orbit is elliptical, the RHS of the above is in the range $(1 - e_o)/2 - (1 + e_o)/2$, depending on the precise value of r when the explosion occurs. In the limit that the initial eccentricity is unity, it is therefore possible both that very little mass loss could disrupt the orbit, if the stars are near apastron, or that the binary could remain stable in spite of losing almost all the binary's mass, if the stars are near periastron.

If the supernova remnant star receives a kick velocity, which is directed at an angle of $\cos\theta$ with respect to $\vec{V}_o$, we find

$$\vec{V} = \vec{V}_o + \Delta\vec{V}, \qquad \vec{V} \cdot \vec{V} = V^2 = V_o^2 + 2V_o\Delta V + (\Delta V)^2 .$$

Therefore, the condition for stability becomes

$$\frac{\Delta M}{M_o} < \frac{r}{2a_o}\left[1 - \frac{\Delta V\,(\Delta V + 2V_o\cos\theta)}{V_c^2}\right].$$

In the case that the kick is exactly (mis)aligned with $\vec{V}_o$, we have that $\cos\theta = +1(-1)$. Comparing the limiting amounts of mass loss for stability in the two cases, denoted ΔM_+ and ΔM_-, respectively, we find

$$\frac{\Delta M_+ - \Delta M_-}{M_o} = -\frac{2r}{a_o}\frac{V_o\Delta V}{V_c^2} < 0.$$

This means that a kick in the same direction as the orbital motion tends to destabilize the orbit, while a misaligned kick stabilizes the orbit. In the case of a circular initial orbit and a misaligned kick, one has

$$\frac{\Delta M}{M_o} = \frac{1}{2}\left[1 + \frac{\Delta V}{V_c^2}\,(2V_c - \Delta V)\right].$$

If the kick velocity magnitude is comparable to the relative orbital speed in this case, one has $\Delta M < M_o$ for stability, i.e., no amount of mass loss can disrupt the orbit! An oppositely directed kick of this size, however, always disrupts the binary.