## The Curve of Growth of the Equivalent Width

Spectral lines are broadened from the transition frequency for a number of reasons. Thermal motions and turbulence introduce Doppler shifts between atoms and the radiation field. The probability that an atom will have a velocity $v$ is

$$\frac{dN}{N} = \frac{e^{-(v/v_0)^2}}{\sqrt{\pi} v_0} dv,$$

where $v_0$ is the mean velocity (of the combined thermal and turbulent motions). The frequency $\nu'$ at which an atom will absorb in terms of the rest frequency $\nu_0$ is

$$\nu' = \nu_0 + \frac{\nu_0 v}{c}.$$

In addition, viewed either classicaly or quantum mechanically, each transition has a damping profile or Lorentz profile, such that the atomic absorption coefficient will be proportional to

$$S_\omega \propto \frac{\Gamma_{ik}}{(\omega - \omega_0)^2 + (\Gamma_{ik}/2)^2}.$$

Here $\Gamma_{ik}$ is related to the Einstein coefficient or strength of spontaneous emission, and $\omega_0$ is the difference in energy of the states. The source of broadening in this case is due to the Heisenberg uncertainty principle. The combined effects in the atomic absorption coefficient are

$$S_\nu (v) \propto \frac{\Gamma_{ik}}{[\nu_0 (1 + v/c) - \nu]^2 + (\Gamma_{ik}/4\pi)^2}.$$

Multiplying this by the probability for the velocity and integrating over all velocities results in

$$S_\nu \propto \Gamma_{ik} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}\nu_0} \frac{e^{-(v/v_0)^2} dv}{[\nu_0 (1 + v/c) - \nu]^2 + (\Gamma_{ik}/4\pi)^2}.$$

Define the dimensionless variables

$$u = \frac{c(\nu - \nu_0)}{\nu_0 v_0}, \qquad y = \frac{v}{v_0}, \qquad a = \frac{c\Gamma_{ik}}{4\pi \nu_0 v_0}.$$

$$S_\nu (u) \propto \frac{a}{\nu_0 v_0} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{a^2 + (u - y)^2} = \frac{\sqrt{\pi}}{\nu_0 v_0} H(a, u).$$

Here the Voigt function is

$$H\left(a,u\right)=\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{a^2+\left(u-y\right)^2}.$$

There are two limiting cases that can be observed. First, for small $a$ and small $u$, the Voigt function behaves as $H(a,u)\rightarrow e^{-u^2}$ since the integrand peaks at $y=u$. The opposite $u\rightarrow\infty$ limit of the Voigt function is

$$H\left(a,u\right)\rightarrow\frac{a}{\pi}\int_{-\infty}^{\infty}u^{-2}e^{-y^2}dy=\frac{au^{-2}}{\sqrt{\pi}}\qquad u\rightarrow\infty,$$

which is valid for any $a$.

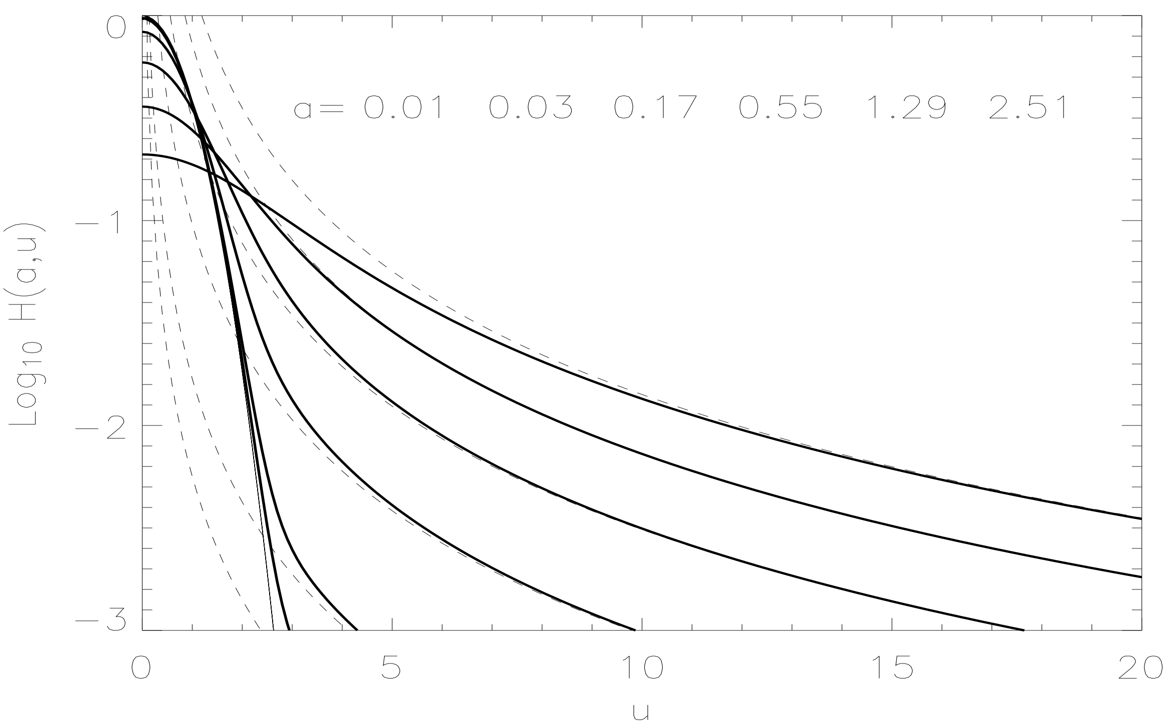


**Figure 1:** The Voigt function $H(a,u)$ for selected values of $a=0.01,,0.5,1.,1.5,2.,2.5,3$. Light solid line indicates the approximation $H(a,u)=e^{-u^2}$ appropriate for $a\approx 0$. Light dashed lines indicate the approximation $H(a,u)=a/(\pi u^2)$ appropriate for large values of $u$.

We can relate the size and shape of the spectral line to the abundance of the species responsible for it. Consider the Schuster-Schwarzschild model, that of a gas layer above the normal atmosphere. In this model, we have

$$r_\nu = \frac{F_\nu}{F_c} = \left(1 + \frac{\sqrt{3}\tau_0}{2}\right)^{-1},$$

where

$$\tau_0 = \int_0^{\tau_0} dt_\nu = \int_0^{z_0} \kappa_\nu \rho dz = \int_0^{z_0} n_i S_\nu dz =< S_\nu > \int_0^{z_0} n_i dz = N_i < S_\nu > .$$

$N_i$ is the column density of the atom giving rise to the line, and $< S_\nu >$ is the line absorption coefficient averaged over depth. Neglecting depth dependences in this model we write $< S_\nu >= S_0 H(a,u)$ and $\chi_0 = S_0 N_i H(a,0)$; the line profile is

$$r_\nu = \left(1 + \frac{\sqrt{3}\tau_0}{2}\right)^{-1} = \left(1 + \frac{\sqrt{3} S_0 H\left(a,u\right) N_i}{2}\right)^{-1} = \left(1 + \frac{\sqrt{3}}{2}\chi_0 \frac{H\left(a,u\right)}{H\left(a,0\right)}\right)^{-1} .$$

The residual flux is illustrated in Fig. 2. Note that for $\chi_0 < 30$, the absorption line is not saturated. For $\chi_0 > 1000$, absorption in the wings of the line is important.

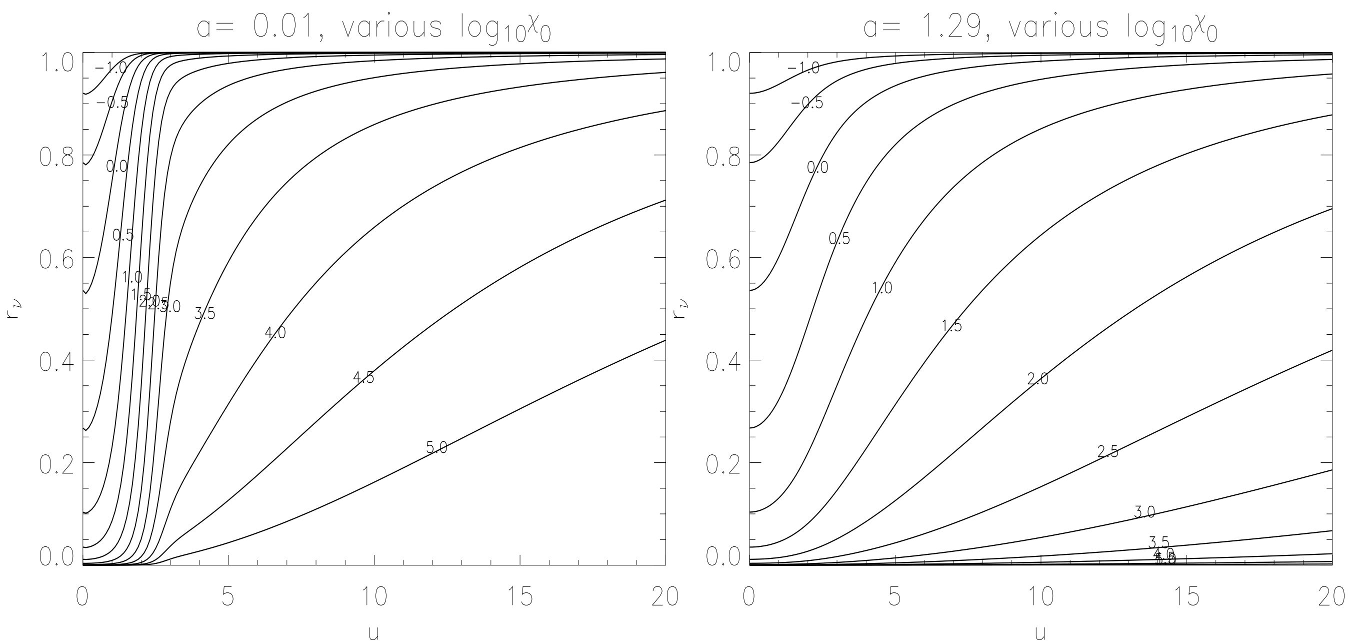


**Figure 2:** Residual flux in the Schuster-Schwarzschild model for an absorption line. Curves are labelled by their values of $\log_{10}\chi_0$. Two values of $a$, 0.01 and 1.29, are illustrated.

Now we can form the equivalent width

$$W_\lambda = \int_{-\infty}^{\infty} \frac{\sqrt{3}\tau_0/2}{1+\sqrt{3}\tau_0/2} d\lambda = 2\Delta\lambda_d \int_0^\infty \frac{\sqrt{3}S_0 H(a,u) N_i du/2}{1+\sqrt{3}S_0 H(a,u) N_i/2},$$

where $du = d\lambda/\Delta\lambda_d$. Using $\chi_0$,

$$\frac{2}{\sqrt{3\pi}} \frac{W_\lambda}{\Delta\lambda_d} = \frac{2\chi_0 H(a,0)}{\sqrt{\pi}} \int_0^\infty \frac{H(a,u)\, du}{H(a,0) + \sqrt{3}\chi_0 H(a,u)/2}.$$

The equivalent width is shown in Fig. 3, as normalized in the above expression, for various values of $a$.

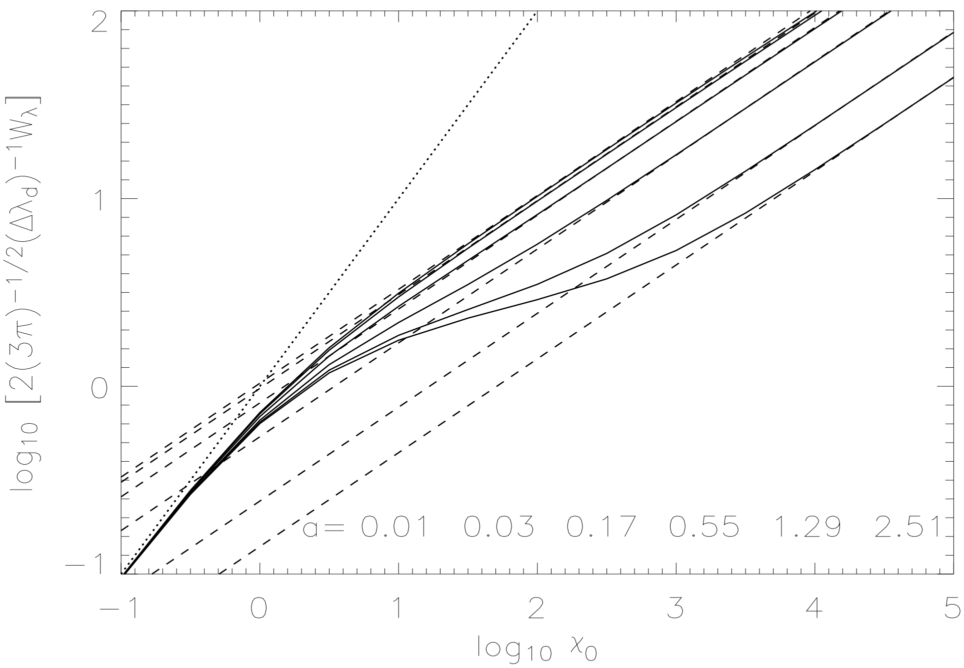


**Figure 3:** Equivalent widths as a function of $a$. The diagonal dotted line represents the small $\chi_0$ limit, while the dashed lines illustrate the limiting behavior for $\chi_0 \to \infty$.

In the limit of weak lines ($a < 1$, moderate $\chi_0$), using $H(a,u) \approx e^{-u^2}$ yields

$$\begin{aligned}\frac{2}{\sqrt{3\pi}} \frac{W_\lambda}{\Delta\lambda_d} &\simeq \frac{4H(a,0)}{\sqrt{3\pi}} \int_0^\infty du \left(1 + \frac{2H(a,0)}{\sqrt{3}\chi_0} e^{u^2}\right)^{-1} \\ &= \frac{2H(a,0)}{\sqrt{3\pi}} \int_0^\infty \frac{dx x^{-1/2}}{1+e^{x-\ln\left(\sqrt{3}\chi_0/2H(a,0)\right)}} \qquad (71) \\ &= \frac{2H(a,0)}{\sqrt{3\pi}} F_{-1/2}\left(\ln\sqrt{3}\chi_0/2H(a,0)\right),\end{aligned}$$

with $F$ the usual Fermi integral. In the limit that $\chi_0 \rightarrow 0$, this becomes

$$\frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} \simeq \chi_0 + \ldots \qquad \chi_0 << 1, \quad a < 1.$$

The equivalent width is proportional to $N_i$, the column density of absorbers. When $\chi_0$ is moderately large, the opposite expansion of $F_{-1/2}$ yields

$$\frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} \simeq \sqrt{\ln\left(\frac{\sqrt{3}\chi_0}{2H\left(a,0\right)}\right)} + \ldots \qquad \chi_0 > 1, \quad a < 1$$

and the line saturates, increasing only as $\sqrt{\ln N_i}$. Note from Fig. 3, that for $a > 1/2$ this intermediate limit is never achieved in practice.

As the number of absorbers grows still further, however, absorption in the wings becomes important. The relevant case is to take the large $u$ limit of $H(a,u) \propto u^{-2}$. $W_\lambda$ will thus grow faster again:

$$\begin{aligned}\frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} &\simeq \frac{4H\left(a,0\right)}{\sqrt{3\pi}}\int_0^\infty du\left(\sqrt{\frac{\pi}{3}}\frac{2H\left(a,0\right)}{a\chi_0}u^2+1\right)^{-1} \\ &= \left(\frac{\pi}{3}\right)^{1/4}\sqrt{2aH\left(a,0\right)\chi_0}, \qquad \chi_0 >> 1\end{aligned}$$

which depends on $\sqrt{N_i}$.

These results are valid for scattering lines, but other applications may require a more sophisticated treatment.