Basic Assumptions for Stellar Atmospheres

A stellar atmosphere is by definition a boundary, one in which photons decouple from matter. Deep in an atmosphere, photons and matter are in strict thermodynamic equilibrium. Near the surface, however, the photon mean free path becomes comparable to the length scale (temperature scale height or pressure scale height) and the photons decouple, eventually becoming freely streaming. Nevertheless, the matter itself is maintained in local thermodynamic equilibrium nearly up to the physical surface itself, by which point the photons are nearly completely decoupled.

It is often useful to discriminate between the continuum and lines in the emergent spectrum, although the continuum is in reality the sum of many weak lines.

The thickness of the atmosphere will generally be very small compared to the radius of the star. Then the geometry will be that of a semi-infinite slab, in which the gravity is constant: $g = GM/R^2$. The equation of hydrostatic equilibrium is then

$$dP/dz = g\rho,$$

where $z = (R - r)$ is the depth. In addition, no appreciable sources or sinks of energy exist in a normal atmosphere, so the conservation of energy is

$$\nabla \cdot \vec{F} = 0 = dF/dz, \qquad F = \text{ constant} = \sigma T_e^4 = L/\left(4\pi R^2\right),$$

where F is the flux and T_e is the effective temperature.

Equation of Radiative Transfer

The flow equation for photons is derived from the Boltzmann transport equation. In general, if f represents the density in phase space (both spatial and momentum), we can write

$$\frac{\partial f}{\partial t} + \sum_i^3 \left(\dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} \right) = S, \tag{1}$$

where S is the source function which governs the creation and destruction, locally, of photons.

The specific intensity describes the flow of energy in a particular direction ($\vec{n}$), through a differential area (dA), into a differential solid angle ($d\Omega$), at a specific point:

$$I_\nu\left(p, \vec{n}\right) = \frac{dE_\nu}{dA \cos\theta d\Omega d\nu dt}.$$

The momentum p is related to the frequency ν by $p = h\nu/c$. The number of photons travelling in direction $\vec{n}$ and crossing dA in a time dt comes from the volume $dV = cdAdt$, while the number of photons occupying that volume is

$$dN = f\left(4\pi p^2 dp\right)(cdAdt).$$

Therefore the energy contained in those "travelling" photons, moving in the direction $\vec{n}$, is

$$dE_\nu = h\nu\cos\theta dN d\Omega/4\pi.$$

Therefore

$$I_\nu(p,\vec{n}) = \frac{h^4\nu^3}{c^3}f, \qquad f = \frac{c^3}{h^4\nu^3}I_\nu(p,\vec{n}).$$

Eq. (1) can be written

$$\frac{\partial f}{\partial t} + \dot{\vec{r}}\cdot\nabla f + \dot{\vec{p}}\cdot\nabla_p f = S,$$

which can be simplified because we will be assuming no time dependence for our atmosphere, and that there are no strong potentials influencing the photons, so that $\dot{\vec{p}} = 0$. Using $\dot{\vec{r}} = c\vec{n}$, and $\vec{n}\cdot\nabla = \cos\theta d/dr$

$$\cos\theta\frac{dI_\nu}{dr} = \frac{h^4\nu^3}{c^3}S.$$

Photons are lost from the flow due to absorption and to scattering, but are added to the flow due to emission and to scattering. This can be summarized by

$$\begin{aligned}\frac{h^4\nu^3}{c^2}S =& \rho j_\nu - (\kappa_\nu + \sigma_\nu)\rho I_\nu(\Omega) + \\ & \frac{\rho\sigma_\nu}{4\pi}\int_0^\infty\int_{4\pi} R_{\nu,\nu'}\left(\Omega,\Omega'\right) I_{\nu'}\left(\Omega'\right) d\Omega' d\nu'.\end{aligned} \tag{2}$$

Here, $R = \frac{h}{c}(\frac{\nu}{\nu'})^3\mathcal{R}$, where R is the scattering redistribution function, normalized so

$$\int_0^\infty\int_0^\infty\int_{4\pi}\int_{4\pi}\mathcal{R}_{\nu',\nu}\left(\Omega',\Omega\right) d\Omega' d\Omega d\nu' d\nu = 1.$$

Also, j_ν is the volume emissivity, κ_ν is the opacity (mass absorption coefficient), and σ_ν is the scattering opacity (mass scattering coefficient). In thermal equilibrium, Kirchoff's law stipulates that

$$j_\nu = \kappa_\nu B_\nu(T)$$

where $B_\nu(T)$ is Planck's function. The optical depth is defined

$$d\tau_\nu = -\left(\kappa_\nu + \sigma_\nu\right)\rho dr = \left(\kappa_\nu + \sigma_\nu\right) dz$$

(note it is frequency dependent) and we may now write the equation of radiative transfer

$$\mu \frac{dI_\nu\left(\mu, \tau_\nu\right)}{d\tau_\nu} = I_\nu\left(\mu, \tau_\nu\right) - S_\nu\left(\mu, \tau_\nu\right). \tag{3}$$

The source function S_ν is the ratio of the total emissivity to the total opacity

$$S_\nu = \frac{\kappa_\nu B_\nu}{\kappa_\nu + \sigma_\nu} + \frac{\sigma_\nu}{4\pi\left(\kappa_\nu + \sigma_\nu\right)} \int_0^\infty \int_{4\pi} R_{\nu',\nu} I_{\nu'}\left(\Omega'\right) d\Omega' d\nu'.$$

To appreciate the meaning of the source function, note that if scattering is unimportant, $S_\nu = B_\nu$ since all photons locally contributed to the radiation field are thermal. If pure absorption processes are negligible, then the source function depends only on the incident radiation field, and is just an average of the specific intensity over angle and energy. In this case, the source function is not dependent on local conditions (i.e., ρ and T). This is the situation in a fog: the light transmitted through the fog carries no information about the physical conditions in the fog. But the transfer equation can be solved without knowing anything about the fog.

The redistribution function can have the following limits:

- Coherent scattering: no energy change, but a directional change. Then R contains $\delta(\nu - \nu')$.
- Noncoherent scattering: frequency of scattered photon is uncorrelated with that of the incident photon. Then R is independent of both ν' and ν.
- Isotropic scattering: direction of scattered photon is uncorrelated with that of incident photon. Then R is independent of both Ω' and Ω.
- Coherent isotropic scattering: $R = \delta(\nu - \nu')$. This is the situation prevailing in normal stellar atmospheres, and one has

$$S_\nu = \frac{\kappa_\nu B_\nu}{\kappa_\nu + \sigma_\nu} + \frac{\sigma_\nu}{4\pi\left(\kappa_\nu + \sigma_\nu\right)} \int_{4\pi} I_\nu\left(\Omega'\right) d\Omega'.$$

Moments of the Radiation Field

Mean Intensity (Zeroth Moment)

$$\begin{aligned} J_\nu(\tau_\nu) &= \frac{\int_{4\pi} I_\nu(\mu,\tau_\nu)\,d\Omega}{\int_{4\pi} d\Omega} = \frac{1}{4\pi}\int_{4\pi} I_\nu(\mu,\tau_\nu)\,d\Omega \\ &= \frac{1}{2}\int_{-1}^{1} I_\nu(\mu,\tau_\nu)\,d\mu, \end{aligned} \tag{4}$$

where the last holds in a plane-parallel atmosphere.

Flux (First Moment)

$$\begin{aligned} \vec{H}_\nu(\tau_\nu) &= \frac{\int_{4\pi} I_\nu(\mu,\tau_\nu)\,\vec{n}d\Omega}{\int_{4\pi} d\Omega} = \frac{1}{4\pi}\int_{4\pi} I_\nu(\mu,\tau_\nu)\,\vec{n}d\Omega \\ &= \frac{\vec{n}}{2}\int_{-1}^{1} I_\nu(\mu,\tau_\nu)\,\mu d\mu \end{aligned} \tag{5}$$

We will use the radiative flux, defined by

$$F_\nu(\tau_\nu) = 2\int_{-1}^{1} I_\nu(\mu,\tau_\nu)\,\mu d\mu. \tag{6}$$

Pressure (Second Moment)

In the plane-parallel case, one can define

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu,\tau_\nu)\,\mu^2 d\mu = \frac{c}{4\pi}P_\nu(\tau_\nu)\,. \tag{7}$$

Radiative Equilibrium

Integrate the radiative transfer equation over all ν and Ω:

$$\frac{1}{\rho}\frac{d}{dz}\int d\nu\int_{4\pi} I_\nu \mu d\Omega = \int d\nu \int_{4\pi}(\kappa_\nu+\sigma_\nu)\,I_\nu d\Omega - \int_{4\pi} d\Omega\int(\kappa_\nu+\sigma_\nu)\,S_\nu d\nu = 0. \tag{8}$$

This vanishes because in local equilibrium the energy gained from the beam must balance the energy lost. This means both

$$\int(\kappa_\nu+\sigma_\nu)\,J_\nu d\nu = \int(\kappa_\nu+\sigma_\nu)\,S_\nu d\nu, \qquad \frac{d}{dz}\int F_\nu d\nu = \frac{dF}{dz} = 0. \tag{9}$$

Moments of the Radiative Transfer Equation

In turn, we multiply the transfer equation by powers of μ, then integrate over μ. Assume coherent isotropic scattering, for which

$$S_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu.$$

Now integrating the transfer equation over μ yields

$$\frac{1}{4}\frac{dF_\nu\left(\tau_\nu\right)}{d\tau_\nu} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}\left[B_\nu\left(\tau_\nu\right) - J_\nu\left(\tau_\nu\right)\right].$$

Scattering contributions have disappeared. Multiply by μ and integrate to obtain the first moment equation:

$$2\frac{dK\left(\tau_\nu\right)}{d\tau_\nu} = \frac{1}{2}F_\nu\left(\tau_\nu\right), \tag{10}$$

as the integral of μS_ν vanishes.

Note that if we integrate the zeroth moment equation over all frequencies, the right-hand side must vanish since no energy is gained or lost in the atmosphere. Then we have

$$\frac{dF\left(\tau\right)}{d\tau} = 0, \qquad F\left(\tau\right) = \text{ constant.} \tag{11}$$

Boundary Conditions

Imagine that one can expand the radiation field:

$$I_\nu\left(\mu, \tau_\nu\right) = \sum_i I_i\left(\tau_\nu\right)\mu^i,$$

which is especially useful when I_0 dominates (isotropy of radiation field). Then, $J_\nu \simeq I_0$, $F_\nu \simeq (4/3)I_1$, $K_\nu \simeq I_0/3$, which will apply deep in the interior. Thus in conditions of near isotropy we have that

$$K_\nu\left(\tau_\nu\right) \simeq \frac{1}{3}J_\nu\left(\tau_\nu\right), \qquad \tau_\nu \to \infty \tag{12}$$

which is known as the diffusion approximation. It can be used to close the moment equations:

$$\frac{dF_\nu\left(\tau_\nu\right)}{d\tau_\nu} = \frac{4\kappa_\nu}{\kappa_\nu + \sigma_\nu}\left[B_\nu\left(\tau_\nu\right) - J_\nu\left(\tau_\nu\right)\right], \quad \frac{dJ\left(\tau_\nu\right)}{d\tau_\nu} = \frac{3}{4}F_\nu\left(\tau_\nu\right). \tag{13}$$

Consider instead conditions near the surface. Generally, there is no incident radiation field. Assuming the emergent intensity nevertheless to be nearly isotropic in the forward direction, we have

$$J_\nu(0) = \frac{1}{2}\int_0^1 I_\nu(\mu, 0)\,d\mu = \frac{1}{2}\sum_i \frac{I_i(0)}{i+1} \simeq \frac{I_0(0)}{2},$$

$$F_\nu(0) = 2\int_0^1 I_\nu(\mu, 0)\,\mu d\mu = 2\sum_i \frac{I_i(0)}{i+2} \simeq I_0(0)\,.$$

Therefore,

$$J_\nu(0) = F_\nu(0)/2, \qquad \tau_\nu \to 0 \tag{14}$$

which is the Eddington approximation.

Solutions of the Radiative Transfer Equation

Classical Solution

For simplicity, we will drop the ν subscript (but remeber that it is there!).

$$\mu\frac{dI(\mu,\tau)}{d\tau} = I(\mu,\tau) - S(\mu,\tau) \tag{15}$$

This equation has an integrating factor $e^{-\tau/\mu}$:

$$\mu\frac{d}{d\tau}\left[I(\mu,\tau)\,e^{-\tau/\mu}\right] = -S(\tau)\,e^{-\tau/\mu}.$$

Integrating:

$$I(\mu,\tau)\,e^{-\tau/\mu} = -\int S(t)\,e^{-t/\mu}\frac{dt}{\mu}, \tag{16}$$

to within a constant. Suppose we evaluate this between two optical depth points, τ_1 and τ_2. Then

$$I(\mu,\tau_1) = I(\mu,\tau_2)\,e^{(\tau_1-\tau_2)/\mu} + \int_{\tau_1}^{\tau_2} S(t)\,e^{(\tau_1-t)/\mu}\frac{dt}{\mu}. \tag{17}$$

The emergent intensity of a semi-infinite slab can be found if we take $\tau_1 = 0$ and $\tau_2 = \infty$:

$$I(\mu,0) = \int_0^\infty S(t)\,e^{-t/\mu}\frac{dt}{\mu}, \tag{18}$$

which is a weighted mean of the source function, the weighting function being the fraction of energy that can penetrate from depth t to the surface. If S is a linear function of depth $S(t) = a + bt$ then $I(\mu, 0)$ is the Laplace transform of S, $I(0, \mu) = a + b\mu$.

Now suppose that we have a finite atmosphere of thickness T within which S is constant. Then the emergent intensity is

$$\begin{aligned} I(\mu, 0) &= I(\mu, T) e^{-T/\mu} + S \int_0^T e^{-t/\mu} \frac{dt}{\mu} \\ &= I(\mu, T) e^{-T/\mu} + S\left(1 - e^{-T/\mu}\right). \end{aligned} \tag{19}$$

If $T >> 1$ we have $I(\mu, 0) = S$: the intensity saturates and is independent of angle.

It is most convenient to discuss this equation at an arbitray point when we impose one of two boundary conditions, either at $\tau = 0$ or $\tau = \infty$. If $\tau_1 = 0$ we have

$$I(\mu, 0) = I(\mu, \tau) e^{-\tau/\mu} + \int_0^\tau S(t) e^{-t/\mu} \frac{dt}{\mu},$$

or

$$I(\mu, \tau) = I(\mu, 0) e^{\tau/\mu} - \int_0^\tau S(t) e^{(\tau - t)/\mu} \frac{dt}{\mu}.$$

In particular, for $\mu < 0$ when there is no incident radiation ($I(\mu, 0) = 0$),

$$I(\mu, \tau) = -\int_0^\tau S(t) e^{(\tau - t)/\mu} \frac{dt}{\mu}. \qquad -1 < \mu < 0 \tag{20}$$

For $\mu > 0$ we can take $\tau_1 = \infty$, on the other hand, and using $\tau_2 = \tau$

$$I(\mu, \tau) = \int_\tau^\infty S(t) e^{(\tau - t)/\mu} \frac{dt}{\mu}. \qquad 1 > \mu > 0 \tag{21}$$

Schwarzschild-Milne Integral Equations

Consider the mean intensity

$$\begin{aligned} J(\tau) &= \frac{1}{2} \int_{-1}^{+1} I(\mu, \tau) \, d\mu \\ &= \frac{1}{2} \int_0^1 d\mu \int_\tau^\infty S(t) e^{(\tau - t)/\mu} \frac{dt}{\mu} - \frac{1}{2} \int_{-1}^0 d\mu \int_0^\tau S(t) e^{(\tau - t)/\mu} \frac{dt}{\mu}. \end{aligned}$$

Interchange the order of integration:

$$J(\tau) = \frac{1}{2}\int_\tau^\infty S(t)\,dt \int_1^\infty e^{-w(t-\tau)}\frac{dw}{w} + \frac{1}{2}\int_0^\tau S(t)\,dt \int_1^\infty e^{-w(\tau-t)}\frac{dw}{w},$$

where we used $w = 1/\mu$ in the first term and $w = -1/\mu$ in the second. The w integrals are called exponential integrals:

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt = x^{n-1}\int_x^\infty t^{-n} e^{-t} dt. \tag{22}$$

Note that

$$E_n'(x) = -E_{n-1}(x) \tag{23}$$

and

$$E_n(x) = \frac{1}{n-1}\left[e^{-x} - xE_{n-1}(x)\right], \qquad n > 1. \tag{24}$$

For large arguments, an asymptotic expansion exists:

$$E_1(x) = \frac{e^{-x}}{x}\left[1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right]. \tag{25}$$

For small x, we can use

$$E_1(x) = -\gamma - \ln x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{kk!}, \qquad x > 0, \tag{26}$$

where $\gamma = 0.5572156\ldots$. Obviously, $E_1(x)$ is singular at the origin, but $E_n(0) = (n-1)^{-1}$ is finite for $n > 1$. However, $E_2(x)$ has a singularity in its first derivative at the origin: $E_2'(0) = -E_1(0)$.

It is useful to collect also these results for the integrals of elementary functions with E_1:

$$\frac{1}{2}\int_0^\infty E_1(|t-\tau|)\, t^p dt = \frac{p!}{2}\left[\sum_{k=0}^{p}\frac{\tau^k}{k!}\delta_\alpha + (-1)^{p+1} E_{p+2}(\tau)\right], \tag{27}$$

where $\delta_\alpha = 0$ if $\alpha = p + 1 - k$ is even, and $\delta_\alpha = 2/\alpha$ if α is odd. For $p = 0;\ 1;\ 2$, the right-hand side of Eq. (27) is $1 - E_2(\tau)/2;\ [\tau + E_3(\tau)/2];\ (2/3 - E_4(\tau) + \tau^2)$. Finally, for $a > 0$ and $a \neq 1$,

$$\frac{1}{2}\int_0^\infty E_1(|t-\tau|)\, e^{-at} dt = \frac{e^{-a\tau}}{2a}\left[\ln\left|\frac{a+1}{a-1}\right| - E_1(\tau - a\tau)\right] + \frac{E_1(\tau)}{2a}.$$

The mean flux can be written now as

$$J(\tau) = \frac{1}{2}\int_{\tau}^{\infty} S(t)\, E_1(t-\tau)\, dt + \frac{1}{2}\int_{0}^{\tau} S(t)\, E_1(\tau - t)\, dt \\ = \frac{1}{2}\int_{0}^{\infty} S(t)\, E_1(|t-\tau|)\, dt. \tag{28}$$

Similarly, we can find

$$F(\tau) = 2\int_{\tau}^{\infty} S(t)\, E_2(t-\tau)\, dt - 2\int_{0}^{\tau} S(t)\, E_2(\tau - t)\, dt, \tag{29}$$

and

$$K(\tau) = \frac{1}{2}\int_{0}^{\infty} S(t)\, E_3(|t-\tau|)\, dt. \tag{30}$$

Recall that the source function, in the case of coherent isotropic scattering, can be written as

$$S = \frac{\kappa}{\kappa+\sigma} B + \frac{\sigma}{\kappa+\sigma} J \equiv J + \epsilon (B - J), \tag{31}$$

so we can find an integral equation for the source function itself:

$$S(\tau) = \epsilon B(\tau) + \frac{1-\epsilon}{2}\int_{0}^{\infty} S(t)\, E_1(|\tau - t|)\, dt. \tag{32}$$

The Planck function makes this equation inhomogeneous. This equation is more general than the assumptions indicate. As long as the angular dependence of the redistribution function is known, it is possible to do the solid angle integrals and express the source function as moments of the radiation field. The moments can be generated from the classical solution, which yields an integral equation like the above.

Since S can be written in terms of J, we also have

$$J(\tau) = \int_{0}^{\infty} \frac{\epsilon}{2} B(t)\, E_1(|\tau - t|)\, dt + \int_{0}^{\infty} \frac{1-\epsilon}{2} J(t)\, E_1(|\tau - t|)\, dt. \tag{33}$$

Remember that ϵ is a function of τ and has to be inside the integrals.

Asymptotic Form of the Transfer Equation

The condition of radiative equilibrium demands that at each point

$$\int_0^\infty (\kappa_\nu + \sigma_\nu) J_\nu(\tau_\nu)\, d\nu = \int_0^\infty (\kappa_\nu + \sigma_\nu) S_\nu(\tau_\nu)\, d\nu.$$

For isotropic coherent scattering, we have

$$\int_0^\infty (\kappa_\nu + \sigma_\nu) J_\nu(\tau_\nu)\, d\nu = \int_0^\infty \kappa_\nu B_\nu(\tau_\nu)\, d\nu + \int_0^\infty \sigma_\nu J_\nu(\tau_\nu)\, d\nu,$$

or simply

$$\int_0^\infty \kappa_\nu J_\nu(\tau_\nu)\, d\nu = \int_0^\infty \kappa_\nu B_\nu(\tau_\nu)\, d\nu. \tag{34}$$

The scattering has cancelled out. This suggests that at large optical depth, the source function is nearly the Planck function. Also, the thermal emission is set by the local radiation field.

Now consider great depths in a semi-infinite atmosphere, where we expect that $S_\nu \approx B_\nu$. Making a Taylor expansion:

$$S_\nu(t) = \sum_{n=0}^\infty \frac{(t-\tau)^n}{n!} \frac{d^n B_\nu(\tau)}{d\tau^n}. \tag{35}$$

Substituting into the classical solution Eq. (16) we find for $\mu > 0$

$$\begin{aligned} I_\nu(\mu,\tau) &= \sum_{n=0}^\infty \frac{1}{n!}\frac{d^n B_\nu}{d\tau^n} \int_\tau^\infty (t-\tau)^n e^{(\tau-t)/\mu} \frac{dt}{\mu} = \sum_{n=0}^\infty \frac{d^n B_\nu}{d\tau^n}\frac{1}{n!}\int_0^\infty x^n e^{-x/\mu}\frac{dx}{\mu} \\ &= \sum_{n=0}^\infty \mu^n \frac{d^n B_\nu}{d\tau^n} = B_\nu(\tau) + \mu\frac{dB_\nu}{d\tau} + \mu^2\frac{d^2 B_\nu}{d\tau^2} + \cdots . \end{aligned} \tag{36}$$

For $\mu < 1$, to within terms of $e^{-\tau/\mu} << 1$, the same result for $I_\nu(\mu,\tau)$ exists. Therefore

$$J_\nu(\tau) = \frac{1}{2}\sum_{n=0}^\infty \frac{d^n B_\nu}{d\tau^n}\int_{-1}^1 \mu^n d\mu = \sum_{n=0}^\infty \frac{1}{2n+1}\frac{d^{2n}B_\nu}{d\tau^{2n}} = B_\nu(\tau) + \frac{1}{3}\frac{d^2 B_\nu}{d\tau^2} + \cdots, \tag{37}$$

$$F_\nu(\tau) = \sum_{n=0}^\infty \frac{4}{2n+3}\frac{d^{2n+1}B_\nu}{d\tau^{2n+1}} = \frac{1}{3}\frac{dB_\nu}{d\tau} + \frac{1}{5}\frac{d^3 B_\nu}{d\tau^3} + \cdots, \tag{38}$$

and

$$K_\nu(\tau) = \sum_{n-0}^{\infty} \frac{1}{2n+3} \frac{d^{2n}B_\nu}{d\tau^{2n}} = \frac{1}{3}B_\nu(\tau) + \frac{1}{5}\frac{d^2B_\nu}{d\tau^2} + \cdots. \tag{39}$$

Note the relation to the diffusion approximation Eq. (12) established earlier. We can write

$$F_\nu = -\frac{4}{3}\left(\frac{1}{\kappa_\nu \rho}\frac{dB_\nu}{dT}\right)\frac{dT}{dr}, \tag{40}$$

where the coefficient of dT/dr is the *radiative conductivity.* This equation is simply the stellar structure luminosity equation established earlier.

It turns out to be conceptually simplifying to keep μ positive for *all* rays. Thus, where $\mu < 0$ in the above, we will write $-\mu$ henceforth. The classical solution, using $\tau_1 = 0, \tau_2 = \tau$ and $I_\nu(-\mu, 0) = 0$ for $\mu < 0$, and $\tau_1 = \tau, \tau_2 = \infty$ for $\mu > 0$ is then

$$\begin{aligned} I_\nu(+\mu,\tau) &= \int_\tau^\infty \frac{S_\nu(t)}{\mu} e^{(\tau-t)/\mu} dt \qquad \mu > 0 \\ I_\nu(-\mu,\tau) &= \int_0^\tau \frac{S_\nu(t)}{\mu} e^{-(\tau-t)/\mu} dt. \qquad \mu < 0 \end{aligned} \tag{41}$$

Mean Opacities

For each a given atmospheric equation, it is possible to write the general frequency-dependent equation in a gray form by defining a different mean opacity. For example, if we wanted a correspondance for the fluxes

$$\int_0^\infty \kappa_\nu F_\nu d\nu \equiv \kappa_F F,$$

we could define a flux-weighted mean:

$$\bar{\kappa}_F = \int_0^\infty \kappa_\nu (F_\nu/F)\, d\nu. \tag{42}$$

However, a practical difficulty is that we don't know F_ν a priori.

A correspondance for the integrated flux

$$\int_0^\infty F_\nu d\nu = F$$

would instead imply the mean $\bar{\kappa}$:

$$\frac{1}{\rho\bar{\kappa}}\frac{dK}{dz} = \frac{F}{4} = \int_0^\infty \frac{F_\nu}{4}d\nu = \int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{dK_\nu}{dz}d\nu. \tag{43}$$

Obviously, we don't know K_ν a priori either, but at great depth, where $3K_\nu \rightarrow J_\nu \rightarrow B_\nu$, we can define the Rosseland mean opacity

$$\frac{1}{\bar{\kappa}_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dz}}{\int_0^\infty \frac{dB_\nu}{dz}d\nu} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}d\nu}{\frac{dB}{dT}}. \tag{44}$$

This choice is especially useful, since the frequency-integrated form of the structure equation Eq. (40) would involve precisely this mean.

Gray Atmospheres

In a gray atmosphere, there is by definition no frequency dependence. The condition of radiative equilibrium then states simply that

$$S(\tau) = J(\tau) = B(\tau) = \frac{\sigma T^4}{\pi}, \tag{45}$$

illustrating that the individual roles of scattering and absorption are irrelevant. The integral equations for the source function and the moments of the radiation field become:

$$B(\tau) = \frac{1}{2}\int_0^\infty B(t)\,E_1(|t-\tau|)\,dt, \qquad J(\tau) = \frac{1}{2}\int_0^\infty J(t)\,E_1(|t-\tau|)\,dt,$$

$$F(\tau) = 2\int_\tau^\infty B(t)\,E_2(t-\tau)\,dt - 2\int_0^\tau B(t)\,E_2(\tau-t)\,dt,$$

$$K(\tau) = \frac{1}{2}\int_0^\infty B(t)\,E_3(|t-\tau|)\,dt. \tag{46}$$

The zeroth moment of the transfer equation is now

$$\frac{1}{4}\frac{dF}{d\tau} = J - J = 0 \tag{47}$$

and the first moment equation is

$$\frac{dK}{d\tau} = \frac{F}{4}. \tag{48}$$

These are integrable:

$$K(\tau) = \frac{1}{4} F\tau + \text{ constant.} \tag{49}$$

At very large depth, the diffusion approximation gives $J(\tau) \to 3K(\tau) \to 3F\tau/4$, and at the surface the Eddington approximation gives $J(0) = F(0)/2$. Therefore, a general expression for J is

$$J(\tau) = \frac{3}{4} F[\tau + q(\tau)]. \tag{50}$$

From Eq. (46) one sees that then

$$\tau + q(\tau) = \frac{1}{2} \int_0^\infty [t + q(t)] E_1(|t - \tau|)\, dt.$$

In addition, the constant in Eq. (49) must be $Fq(\infty)/4$ since $K = 3J$ as $\tau \to \infty$. A general solution of the gray atmosphere is equivalent to solving for $q(\tau)$. We will look at some approximate solutions.

Approximate Solutions

Evaluating the expression Eq. (50) at the surface, we find

$$J(0) = \frac{3F}{4} q(0) \simeq \frac{F}{2},$$

or $q(0) \simeq 2/3$. The simplest solution to the gray atmosphere problem is simply to choose $q(\tau) = 2/3$, often called the *Eddington approximation.*

Formally, the Eddington approximation consists of assuming $K = J/3$ everywhere. This has already shown to be true at great depths, but in fact is true in a wider variety of situations also. Consider:

a) $I(\mu)$ expandable in odd powers of μ only (except for I_0 which is still dominant). Therefore only the I_0 term contributes to J or K and we generally obtain $J = 3K$.

b) $I(\mu) = I_0$ for $\mu > 0$ and 0 for $\mu < 0$. Then

$$J = \frac{I_0}{2} \int_0^1 d\mu = \frac{I_0}{2}, \qquad K = \frac{I_0}{2} \int_0^1 \mu^2 d\mu = \frac{I_0}{6}.$$

c) Two-stream model. $I(\mu) = I_+$ for $\mu > 0$ and I_- for $\mu < 0$. Then

$$\begin{aligned} J =& \frac{I_+}{2} \int_0^1 d\mu + \frac{I_-}{2} \int_{-1}^0 d\mu = \frac{I_+ + I_1}{2}, \\ K =& \frac{I_+}{2} \int_0^1 \mu^2 d\mu + \frac{I_-}{2} \int_{-1}^0 \mu^2 d\mu = \frac{I_+ + I_-}{6}. \end{aligned}$$

An exception is provided by a beam, in which $I(\mu) = \delta(\mu - \mu_0)$, for which $J = I_0$ and $K = I_0\mu_0^2$.

When $J = 3K$, we can write the result $K_E = F\tau/4 + C$ as

$$J_E\left(\tau\right) = B\left(\tau\right) = 3F\tau/4 + C'. \tag{51}$$

Using Eq. (46) for the flux at the surface, we have

$$F\left(0\right) = 2\int_0^\infty \left(\frac{3}{4}Ft + C'\right) E_2\left(t\right) dt = 2C'E_3\left(0\right) + \frac{3}{4}F\left[\frac{4}{3} - 2E_4\left(0\right)\right] = F.$$

With $E_n(0) = (n-1)^{-1}$, we find $C' = F/2$ and

$$J_E\left(\tau\right) = \frac{3}{4}F\left(\tau + \frac{2}{3}\right).$$

Since $B(\tau) = \sigma T^4/\pi$, we also have

$$T^4 = \frac{3}{4}T_{eff}^4\left(\tau + \frac{2}{3}\right),$$

so $T(0) = 2^{-1/4}T_{eff}$. Also note that when $\tau = 2/3$ that $T = T_{eff}$, so the effective depth of the continuum is often taken to be at optical depth 2/3.

Limb Darkening

From the gray result for the mean flux in the Eddington approximation, Eq. (51), we can immediately calculate the angular dependence of the emergent intensity using the classical solution's properties of Laplace transforms:

$$I_E\left(\mu, 0\right) = \frac{3}{4}F\int_0^\infty \left(\tau + \frac{2}{3}\right) e^{-\tau/\mu}\frac{d\tau}{\mu} = \frac{3}{4}F\left(\mu + \frac{2}{3}\right).$$

Applied to the Sun, the center of its disc is at $\mu = 1$. The relative intensity as one traverses the Sun's disc is therefore

$$\frac{I_E\left(\mu, 0\right)}{I_E\left(1, 0\right)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right),$$

with an intensity of the Sun's limb only 40% of its center. This is not in serious disagreement with observations.

Since the source function is determined by T, the depth dependence of T can be determined by measuring the angular dependence of limb darkening. Measurements of this limb darkening therefore yields information on the temperature gradient underneath the surface.

Improvements to Eddington Approximation

Note that in the Eddington approximation $J_E(0) = F/2$ and $I_E(0,0) = F/2$ so that

$$J_E(0) = I_E(0,0).$$

Actually, this result is true in general.

A check on the accuracy of the Eddington approximation is to evaluate

$$J(0) = \frac{1}{2}\int_0^1 I_E(\mu,0)\,d\mu = \frac{1}{2}\int_0^1 \int_0^\infty J_E(t)\,e^{-t/\mu}\frac{dt}{\mu}d\mu = \frac{7}{16}F.$$

Thus, the Eddington approximation is internally not self-consistent.

We can improve upon the Eddington approximation by using Eq. (46): on the right-hand side of the second equation, use the Eddington approximation. Thus

$$J_{E'}(\tau) \simeq \frac{3}{4}F\left[\tau + \frac{1}{2}E_3(\tau) + \frac{2}{3} - \frac{1}{3}E_2(\tau)\right].$$

Asymptotically, this approaches J_E at large depth. The biggest difference between this and J_E occurs at the surface: $J_{E'}(0)/J_E(0) = 7/8$. The new estimate of $T(0)/T_{eff} = (7/16)^{1/4}$, while $q(\infty)$ remains 2/3, but $q(0) = 7/12$ instead of 2/3. (The exact value is $1/\sqrt{3}$, only 1% different). This new estimate for J can be used for an improved estimate of limb darkening:

$$\begin{aligned} I_{E'}(\mu,0) &= \frac{3}{4}F\int_0^\infty e^{-t/\mu}\left[t + \frac{2}{3} + \frac{1}{2}E_3(t) - \frac{1}{3}E_2(t)\right]\frac{dt}{\mu} \\ &= \frac{3}{4}F\left[\frac{7}{12} + \frac{\mu}{2} + \left(\frac{\mu}{3} + \frac{\mu^2}{2}\right)\ln\left(\frac{1+\mu}{\mu}\right)\right]. \end{aligned}$$

We now have $I_{E'}(0,0) = J_{E'}(0) = 7F/16$, and $I_{E'}(0,0)/I_{E'}(0,1) = 0.351$ (the exact value is 0.344). Note that the Eddingon approximation establishes J_E from the assumption that F is constant. Using the third of Eq. (46), one can show that

$$F_E(\tau) = \frac{3}{4}F\left[\frac{4}{3} - 2E_4(\tau) + \frac{4}{3}E_3(\tau)\right],$$

Table 1: Points and Weights for Gauss-Laguerre Quadrature

n	x_i	W_i	n	x_i	W_i
2	0.585786	0.853553	3	0.415775	0.711093
	3.41421	0.146447		2.29428	0.278518
				6.28995	0.0103893
4	0.322548	0.603154	5	0.26356	0.521756
	1.74576	0.357419		1.4134	0.398667
	4.53662	0.0388879		3.59643	0.0759424
	9.39507	0.000539295		7.08581	0.00361176
				12.6408	0.00002337

which is only approximately constant (to within 3%). The result for $F_{E'}$ by using the improvement $J_{E'}$ is about 10 times better.

Another way of solving the gray atmosphere involves the Milne integral relations Eq. (46) themselves. The solution of these equations is not analytic, and care must be taken because of the bad behavior of $E_1(x)$ as $x \to 0$. Some gain is made by adding and subtracting $B(\tau)$ to the right-hand side of the first of Eq. (46):

$$B(\tau) = \frac{1}{2}\int_0^\infty \left[B(t) - B(\tau)\right] E_1(|t-\tau|)\, dt + \frac{B(\tau)}{2}\int_0^\infty E_1(|t-\tau|)\, dt.$$

The integrand of the first of these is well behaved since $B(t) - B(\tau)$ goes to zero faster than the logarithmic divergence of E_1. The second integral follows from the properties of exponential integrals:

$$\int_0^x E_1(x)\, dx = E_2(0) - E_2(x), \qquad E_2(0) = 1, \tag{52}$$

so that we find the well-behaved result

$$B(\tau) = E_2^{-1}(\tau)\int_0^\infty \left[B(t) - B(\tau)\right] E_1(|t-\tau|)\, d\tau. \tag{53}$$

These integrals are efficiently performed using Gauss-Laguerre quadrature

$$B(\tau) = E_2^{-1}(\tau)\sum_{i=0}^{n} \left[B(t_i) - B(\tau)\right] E_1(|t_i-\tau|)\, W_i, \tag{54}$$

where t_i and W_i are the points and weights of the quadrature (see Table 1).

Table 2: Points and Weights for Gauss-Legendre Quadrature

n even	μ_i	a_i	n odd	μ_i	a_i
1	$\pm\frac{1}{\sqrt{3}}$	1	1	0	$\frac{8}{9}$
				$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
2	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{1}{2}+\frac{1}{6}\sqrt{\frac{5}{6}}$	2	0	$\frac{128}{225}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{1}{2}-\frac{1}{6}\sqrt{\frac{5}{6}}$		$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
				$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$
3	±0.23861918	0.46791393	3	0	0.41795918
	±0.66120939	0.36076157		±0.40584515	0.38183005
	±0.93246951	0.17132449		±0.74153119	0.27970539
				±0.94910791	0.12948497
4	±0.18343464	0.36268378	5	±0.14887434	0.29552422
	±0.52553241	0.31370665		±0.43339539	0.26926672
	±0.79666648	0.22238103		±0.67940957	0.21908636
	±0.96028986	0.10122854		±0.86506337	0.14945135
				±0.97390653	0.06667134

Evaluating Eq. (54) at the quadrature points t_i, then rearranging,

$$\sum_{k=1}^{m} B\left(t_k\right)\left[\sum_{i=1}^{n} \frac{\left(\delta_{ik}-\delta_{jk}\right) W_i E_1\left(\left|t_i-t_j\right|\right)}{E_2\left(t_j\right)}-\delta_{kj}\right]=0. \tag{55}$$

These represent n linear homogeneous algebraic equations, an eigenvalue problem. The eigenvalue is the total radiative flux, which is a constant.

Method of Discrete Ordinates

For a gray atmosphere

$$\mu \frac{dI}{d\tau}=I-\frac{1}{2} \int_{-1}^{1} I\left(\mu, \tau\right) d\mu=I-\frac{1}{2} \sum_{j=-n}^{n} a_j I_j. \tag{56}$$

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{1}{2} \sum_{j=-n}^{n} a_j I_j. \tag{57}$$

Here, i, j are $\pm 1, \ldots, \pm n$, and $I_i(\tau) = I(\mu_i, \tau)$. The a_i are the Gauss-Legendre weights for the points μ_i (see Table 2, for n even). We will not use schemes with points where $\mu_i = 0$, see below. We've replaced the continuous radiation field by a finite set of pencil beams. This should become exact in the limit $n \to \infty$. Eq. (57) is a first-order, linear equation. Use a trial function $I_i = g_i e^{-k\tau}$: then we must have

$$g_i = \frac{C}{1 + k\mu_i}, \qquad 1 = \sum_{j=-n}^{n} \frac{a_j}{1 + k\mu_j}. \tag{58}$$

The latter is called the characteristic equation for k. Since $a_{-j} = a_j$ and $\mu_{-j} = -\mu_j$, we can write this as

$$1 = \sum_{j=1}^{n} \frac{a_j}{1 - k^2 \mu_j^2}.$$

Since $\int_{-1}^{1} d\mu = 2$, we have $\sum_{j=-n}^{n} a_j = 2$ and $\sum_{j=1}^{n} a_j = 1$. Thus $k^2 = 0$ is a solution, and there are $n - 1$ additional solutions. These solutions satisfy

$$\frac{1}{\mu_1^2} < k_1^2 < \frac{1}{\mu_2^2} < \cdots < k_{n-1}^2 < \frac{1}{\mu_n^2}.$$

The general solution is then

$$I_i(\tau) = \sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + k_\alpha \mu_i} + \sum_{\alpha=1}^{n-1} \frac{L_{-\alpha} e^{k_\alpha \tau}}{1 - k_\alpha \mu_i}.$$

There is also a particular solution corresponding to the case $k^2 = 0$: Substituting $I_i = b(\tau + q_i)$ into the original differential equation Eq. (57),

$$I_i(\tau) = b(\tau + Q + \mu_i).$$

Thus the complete solution is

$$I_i(\tau) = b(\tau + Q + \mu_i) + \sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + k_\alpha \mu_i} + \sum_{\alpha=1}^{n-1} \frac{L_{-\alpha} e^{k_\alpha \tau}}{1 - k_\alpha \mu_i}.$$

There are still $2n$ unknown coefficients (Q, b and $L_{\pm\alpha}$) to be determined. Use the boundary conditions to do this. In the case of a semi-infinite atmosphere, we have the boundary condition that $I_{-i}(0) = 0$ and also that $I(\tau)$ should remain finite in the limit $\tau \to \infty$. The latter constraint immediately implies that $L_{-\alpha} = 0$. The former constraint means that

$$Q + \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - k_\alpha \mu_i} - \mu_i = 0.$$

Finally, we must demand that the flux equals the flux F, or

$$F = 2 \int_{-1}^{1} \mu I(\mu, \tau)\, d\mu = 2 \sum_{j=-n}^{n} a_j \mu_j I_j(\tau).$$

Using the complete solution above, we have

$$F = 2b \left[(\tau + Q) \sum_{j=-n}^{n} a_j \mu_j + \sum_{j=-n}^{n} a_j \mu_j^2 + \sum_{n=1}^{n-1} L_\alpha e^{-k_\alpha \tau} \sum_{j=-n}^{n} \frac{a_j \mu_j}{1 + k_\alpha \mu_j} \right].$$

Now the first sum is zero, the second sum is $2/3$, and the third sum is

$$\frac{1}{k_\alpha} \sum_{j=-n}^{n} a_j \left(1 - \frac{1}{1 + k_\alpha \mu_j} \right) = \frac{2}{k_\alpha} \left(1 - \frac{1}{2} \sum_{j=-n}^{n} \frac{a_j}{1 + k_\alpha \mu_j} \right) = 0,$$

because of the characteristic equation Eq. (58). So we have that $b = 3F/4$, and a constant flux is then automatic. The final result is

$$I_i(\tau) = \frac{3}{4} F \left(\tau + Q + \mu_i + \sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + k_\alpha \mu_i} \right). \qquad (59)$$

The mean intensity is

$$\begin{aligned} J(\tau) &= \frac{1}{2} \sum_{j=-n}^{n} a_j I_j \\ &= \frac{3}{4} F \left[(\tau + Q) \frac{1}{2} \sum_{j=-n}^{n} a_j + \frac{1}{2} \sum_{j=-n}^{n} a_j \mu_j + \sum_{\alpha=1}^{n-1} L_\alpha e^{-k_\alpha \tau} \frac{1}{2} \sum_{j=-n}^{n} \frac{a_j}{1 + k_\alpha \mu_j} \right]. \end{aligned}$$

With the characteristic equation, this becomes

$$B(\tau) = J(\tau) = \frac{3}{4}F\left[\tau + Q + \sum_{\alpha=1}^{n-1} L_\alpha e^{-k_\alpha \tau}\right].$$

The Hopf function is then

$$q(\tau) = Q + \sum_{\alpha=1}^{n-1} L_\alpha e^{-k_\alpha \tau}, \qquad q(\infty) = Q. \tag{60}$$

For the cases of small n, the solutions for $q(\tau)$ are straightforward:

$$\begin{aligned}
n = 1: &\quad q(\tau) = 1/\sqrt{3} \\
n = 2: &\quad = 0.694025 - 0.116675e^{-1.97203\tau} \\
n = 3: &\quad = 0.703899 - 0.101245e^{-3.20295\tau} - 0.02530e^{-1.22521\tau} \\
n = 4: &\quad = 0.70692 - 0.08392e^{-4.45808\tau} - 0.03619e^{-1.59178\tau} - 0.00946e^{-1.10319\tau}.
\end{aligned}$$

The exact result for $q(0)$ is $1/\sqrt{3}$; only the case $n = 1$ yields this value. The exact result for Q is 0.710446, and the case $n = 1$ gives a value 25% too small. For $n = 4$ the maximum error in J compared to the exact result is about 4%.

One can greatly improve the accuracy of this scheme by recognizing that with no incident radiation, the solution for I has a discontinuity for $\mu = 0, \tau = 0$. Splitting the integral of Eq. (56), which has the discontinuous integrand, into two parts that avoid the discontinuity, and then performing each integral by Gauss-Legendre quadrature, accomplishes this. We have

$$\mu\frac{dI}{d\tau} - I = -\frac{1}{2}\left[\int_{-1}^{0} I(\mu)\,d\mu + \int_{0}^{1} I(\mu)\,d\mu\right] = -\frac{1}{4}\left[\int_{-1}^{1} I(\nu)\,d\nu + \int_{-1}^{1} I(\omega)\,d\omega\right], \tag{61}$$

where

$$\nu = 2\mu + 1, \qquad \omega = 2\mu - 1. \tag{62}$$

Thus

$$\mu_i\frac{dI_i}{d\tau} = I_i - \frac{1}{4}\sum_{j=-n}^{n} a_j\left[I_{\nu_j} + I_{\omega_j}\right]. \tag{63}$$

Formally, this looks identical to Eq. (57) (for an even number of points) if $n \to 2n$, $a_j \to a_j/2$, and the points are accordingly redefined as in

Table 3: Points and Weights for Double-Gauss Quadrature

n	μ_i	a_i
2	$\pm\frac{1}{2}(1 \pm \frac{1}{\sqrt{3}})$	$\frac{1}{2}$
4	$\pm\frac{1}{2}(1 \pm \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}})$	$\frac{1}{4} + \frac{1}{12}\sqrt{\frac{5}{6}}$
	$\pm\frac{1}{2}(1 \pm \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}})$	$\frac{1}{4} - \frac{1}{12}\sqrt{\frac{5}{6}}$

Eq. (62). In short, we use the points and weight in Table 3. The double-Gauss quadrature formula achieves 0.6% accuracy even for $n = 4$. For $n = 2$, one can show that

$$\begin{aligned} I_i(\tau) &= \frac{3}{4}F\left[\tau + Q + \mu_i + \frac{Le^{-k\tau}}{1 + k\mu_i}\right] \\ B(\tau) = J(\tau) &= \frac{3}{4}F\left[\tau + Q + Le^{-k\tau}\right] \\ k &= \sqrt{\frac{1 - a_2}{\mu_2^2} + \frac{1 - a_1}{\mu_1^2}} = 2\sqrt{3} \\ Q &= \mu_1 + \mu_2 - \frac{1}{k} = 1 - \frac{1}{2\sqrt{3}} \\ L &= \frac{(1 - k\mu_1)(1 - k\mu_2)}{k} = \frac{\sqrt{3}}{2} - 1. \end{aligned} \tag{64}$$

Note that this gives an exact result for $q(0)$, and a result for $q(\infty) = Q$ in error by only 0.1%. The discrete ordinate method can be generalized to yield the exact solution, but we won't work it out here.

The Emergent Flux from a Gray Atmosphere

Although in a gray atmosphere the opacity is independent of frequency, the flux dependence on frequency still varies with depth. We have

$$B(\tau) = \frac{\sigma T(\tau)^4}{\pi} = J(\tau), \qquad T(\tau)^4 = \frac{3}{4}T_{eff}^4\left[\tau + q(\tau)\right].$$

From the frequency dependence of the source function, Eq. (46) yields

$$F_\nu(\tau) = 2\int_\tau^\infty B_\nu\left[T(t)\right]E_2(t - \tau)\,dt - 2\int_0^\tau B_\nu\left[T(t)\right]E_2(\tau - t)\,dt,$$

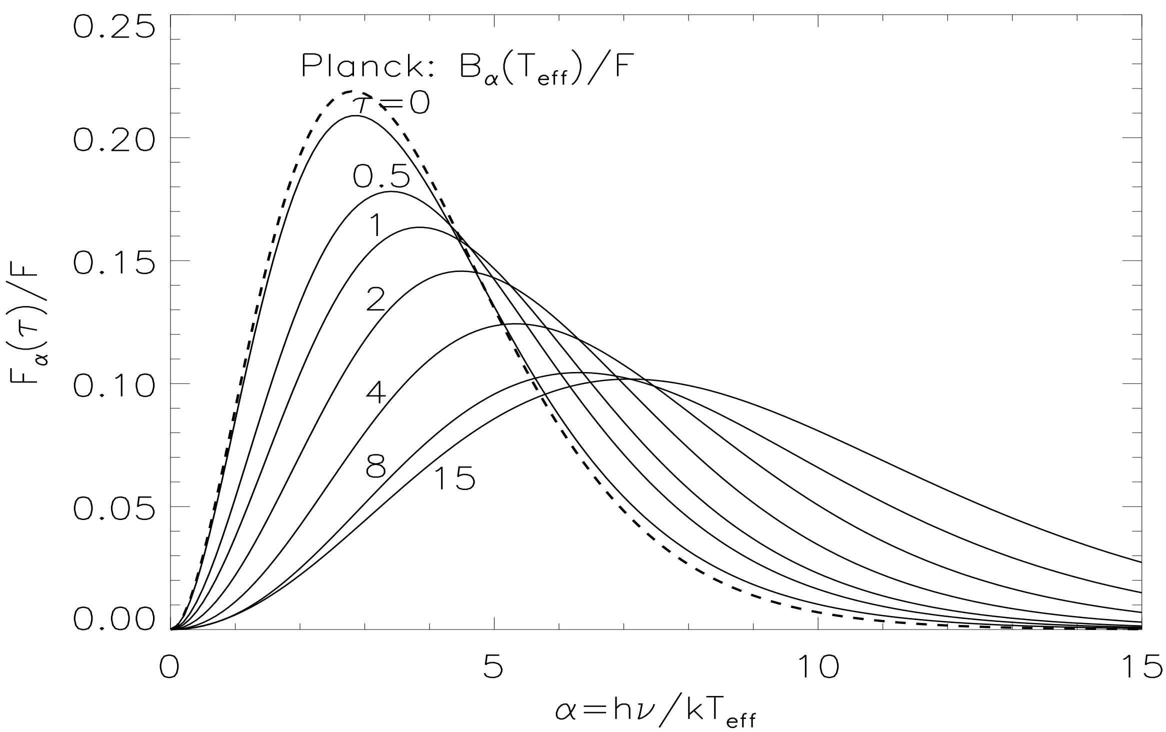
where the Planck function is

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1}.$$

Using the parameter $\alpha = h\nu/kT_{eff}$, where $T_{eff}/T = (3[t + q(t)]/4)^{-1/4}$, the flux is $F_\alpha(\tau) = F_\nu(\tau)\frac{d\nu}{d\alpha}$:

$$\frac{F_\alpha(\tau)}{F} = \frac{30\alpha^3}{\pi^4}\left[\int_\tau^\infty \frac{E_2(t-\tau)\,dt}{e^{\alpha T_{eff}/T} - 1} - \int_0^\tau \frac{E_2(\tau - t)\,dt}{e^{\alpha T_{eff}/T} - 1}\right].$$

As the figure shows, for $\tau = 0(2)$, this peaks near $\alpha = 3(5)$, and the peak value for $\tau = 2$ is 25% smaller than for $\tau = 0$. The mean photon energy is degraded as they are transferred from the interior to the surface. The Planck function (B_α/F) for $T = T_{eff}$ is shown for comparison: the emergent spectra ($\tau = 0$) is slightly harder.



Correction for Stimulated Emission

In general, there are 3 types of transitions:

- Spontaneous Emission: $N_{i\to j} = N_i A_{ij} dt$
- (Stimulated) Absorption $N_{j\to i} = N_j B_{ji} I_{\nu_{ij}} dt$
- Stimulated Emission (enhanced in the presence of a photon of the same energy as the spontaneous transition) $N_{i\to j} = N_i B_{ij} I_{\nu_{ij}} dt$

Note the symmetric process of spontaneous absorption cannot occur. In strict thermal equilibrium, detailed balance occurs, and the photon distribution is the Planck function, so

$$N_j B_{ji} B_{\nu_{ij}}(T) = N_i \left[A_{ij} + B_{ij} B_{\nu_{ij}}(T) \right].$$

The Boltzmann formula must hold for the relative abundances of the two states:

$$\frac{N_i}{N_j} = \frac{g_i}{g_j} e^{-h\nu_{ij}/kT}.$$

Writing out the Planck function:

$$A_{ij}\frac{g_i}{g_j} = \frac{2h\nu^3}{c^2} B_{ji} \frac{e^{h\nu_{ij}/kT} - \frac{B_{ij}g_i}{B_{ji}g_j}}{e^{h\nu_{ij}/kT} - 1}.$$

The Einstein coefficients are independent of temperature (properties of atoms), which can only happen if

$$\frac{B_{ij}g_i}{B_{ji}g_j} = 1, \qquad A_{ij} = B_{ji}\left(\frac{2h\nu^3}{c^2}\frac{g_j}{g_i}\right).$$

Now recall from an early discussion that the source function, in the absence of scattering, is the ratio of the emissivity j_ν to the opacity k_ν. The total energy produced per unit volume and flowing through a solid angle $d\Omega$ is

$$j_\nu \rho d\nu d\Omega = h\nu N_i \left(A_{ij} + B_{ij} I_\nu\right) = N_i A_{ij} h\nu \left(1 + \frac{I_\nu c^2}{2h\nu^3}\right)$$

and the total absorbed energy is

$$I_\nu \kappa_\nu \rho d\nu d\Omega = N_j B_{ji} I_\nu h\nu.$$

Then

$$S_\nu = \frac{j_\nu}{\kappa_\nu} = \frac{N_i A_{ij}\left(1 + \frac{c^2 I_\nu}{2h\nu^3}\right)}{N_j B_{ji}} = \frac{N_i g_j}{N_j g_i}\left(\frac{2h\nu^3}{c^2} + I_\nu\right).$$

$$S_\nu = e^{-h\nu/kT}\left(\frac{2h\nu^3}{c^2} + I_\nu\right) = B_\nu\left(1 - e^{-h\nu/kT}\right) + I_\nu e^{-h\nu/kT}.$$

In the equation of radiative transfer, we have

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu = (I_\nu - B_\nu)\left(1 - e^{-h\nu/kT}\right),$$

which can be turned into

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - B_\nu$$

if the opacity κ_ν is redefined as $\kappa_\nu(1 - e^{-h\nu/kT})$.

Formation of Spectral Lines

Definitions:

$$f_\nu(\mu) = \frac{I_\nu(\mu, 0)}{I_c(\mu, 0)} : \quad \text{residual intensity}$$

$$r_\nu = \frac{F_\nu(0)}{F_c(0)} : \quad \text{residual flux}$$

$$W_\lambda = \int_0^\infty (1 - r_\lambda)\, d\lambda : \quad \text{equivalent width}$$

The subscript ν refers to the line, and the subscript c refers to the continuum. The equivalent width is the width of a completely black line that absorbs the same number of photons as the spectral line of interest. The integrals range of 0 and ∞ just means "far from the line center". Note that $W_\nu \approx (\nu/\lambda) W_\lambda$.

Spectral lines are of two types: pure absorption where the absorbed energy is fully shared with the gas, and resonance lines in which it is not. In the former, the emission of photons is completely uncorrelated with previous absorption. In resonance scattering, the emitted photon is completely correlated with the absorbed photon (coherent scattering). Treating the line and continuum processes separately, the radiative transfer equation is

$$\mu \frac{dI_\nu(\mu, \tau_\nu)}{d\tau_\nu} = I_\nu(\mu, \tau_\nu) - \frac{(\kappa + \kappa_\nu) B_\nu + (\sigma + \sigma_\nu) J_\nu}{\kappa + \kappa_\nu + \sigma + \sigma_\nu},$$

where the optical depth in the line is $d\tau_\nu = (\kappa + \kappa_\nu + \sigma + \sigma_\nu)\rho dz$.

Schuster-Schwarzschild Model

Suppose we have strong resonance lines formed in a thin layer overlying the photosphere. Then $\kappa << \sigma$ and $\sigma_\nu >> \sigma$. Then

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - J_\nu, \qquad d\tau_\nu = -\sigma_\nu \rho dx.$$

This looks like the transfer equation for a gray atmosphere, and we must have from radiative equilibrium $F_\nu(\tau_\nu) =$ constant for each frequency. From the results for a gray atmosphere, using $n = 1$,

$$I_+(\tau_\nu) = \frac{3F_\nu\left(\tau_\nu + 1/\sqrt{3} + Q\right)}{4}, \qquad I_-(\tau_\nu) = \frac{3F_\nu\left(\tau_\nu - 1/\sqrt{3} + Q\right)}{4}.$$

The boundary condition $I_-(0) = 0$ implies that $Q = 1/\sqrt{3}$ and

$$I_+(\tau_\nu) = \frac{3F_\nu\left(\tau_\nu + 2/\sqrt{3}\right)}{4}, \qquad I_-(\tau_\nu) = \frac{3F_\nu\tau_\nu}{4}.$$

If we require that the line intensity on the base of the thin cool gas layer be the same as the emergent intensity of the continuum,

$$I_+(\tau_o) = \frac{3F_c\left(0 + 2/\sqrt{3}\right)}{4} = \frac{3F_\nu\left(\tau_o + 2/\sqrt{3}\right)}{4}.$$

The residual flux is just

$$r_\nu = \frac{F_\nu}{F_c} = \left(1 + \frac{\sqrt{3}\tau_o}{2}\right)^{-1}.$$

The angular dependence can be found from the classical solution

$$I_\nu(\mu, 0) = \int_0^\infty \frac{J_\nu(t_\nu)\, e^{-t_\nu/\mu} dt_\nu}{\mu} + I_c(\mu, 0)\, e^{-\tau_o/\mu}.$$

The mean intensity can be approximated as

$$J_\nu(\tau_\nu) = \frac{1}{2}\left[I_+(\tau_\nu) + I_-(\tau_\nu)\right] = \frac{3F_\nu\left(\tau_\nu + 1/\sqrt{3}\right)}{4}.$$

Using this relations, one finds

$$f_\nu(\mu) = \frac{3F_c}{4I_c(\mu, 0)\left(1 + \sqrt{3}\tau_o/2\right)}\left[\mu + \frac{1}{\sqrt{3}} - \left(\mu + \tau_o + \frac{1}{\sqrt{3}}\right) e^{-\tau_o/\mu}\right] + e^{-\tau_o/\mu}.$$

In the limit of weak lines, $\tau_o << 1$, we find

$$f_\nu(\mu) \simeq 1 - \frac{3F_c}{4I_c(\mu, 0)}\tau_o.$$

There is no angular dependence except what arises due to limb-darkening of the continuum. Thus scattering lines are visible at all point on the stellar disk with roughly equal strength. In the limit of strong lines, $\tau_o >> 1$,

$$f_\nu(\mu) \simeq \frac{\sqrt{3}F_c}{2I_c(\mu,0)\,\tau_o}\left(\mu + 1/\sqrt{3}\right).$$

The range in line strength between the center of the disk and the edge is about 2. This will contrast with that to be found from pure absorption lines, discussed next.

Milne-Eddington Model

In the case of pure absorption, we have to specify something about the depth dependence of the opacity and source function, which was unecessary in the scattering case. Define

$$\epsilon_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}, \qquad \eta_\nu = \frac{\kappa_\nu + \sigma_\nu}{\kappa}, \qquad \mathcal{L}_\nu = \frac{\kappa_\nu + \kappa}{\kappa + \kappa_\nu + \sigma_\nu} = \frac{1 + \eta_\nu \epsilon_\nu}{1 + \eta_\nu}.$$

ϵ_ν measures the importance of absorption to total extinction in the line; η_ν measures the line strength; $\mathcal{L}_\nu$ measures net effect of absorption in line and continuum. The line transfer equation is

$$\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - \mathcal{L}_\nu B_\nu - (1 - \mathcal{L}_\nu)\,J_\nu, \qquad d\tau_\nu = (\kappa + \kappa_\nu + \sigma_\nu)\,\rho dz. \tag{65}$$

Note in the continuum,

$$d\tau = \kappa\rho dz = \frac{\kappa d\tau_\nu}{\kappa + \kappa_\nu + \sigma_\nu} = \frac{d\tau_\nu}{1 + \eta_\nu},$$

so $\tau = \tau_\nu/(1 + \eta_\nu)$. In the Eddington approximation, $B(\tau) = a + b\tau$, or

$$B_\nu(\tau_\nu) = a + b\tau_\nu/(1 + \eta_\nu). \tag{66}$$

Attempt to solve Eq. (65) by taking moments:

$$\frac{dF_\nu}{d\tau_\nu} = 4\mathcal{L}_\nu(J_\nu - B_\nu), \qquad \frac{dK_\nu}{d\tau_\nu} = \frac{F_\nu}{4}.$$

With $K_\nu \approx J_\nu/3$,

$$\frac{d^2 J_\nu}{d\tau_\nu^2} = \frac{3}{4}\frac{dF_\nu}{d\tau_\nu} = 3\mathcal{L}_\nu(J_\nu - B_\nu).$$

Using the linear relation in Eq. (66), we must have

$$J_\nu(\tau_\nu) - B_\nu(\tau_\nu) = ce^{-\sqrt{3\mathcal{L}_\nu}\tau_\nu},$$

where the positive exponent term vanishes since $J_\nu \to B_\nu$ as $\tau_\nu \to \infty$. At the surface, the Eddington approximation leads to $J_\nu(0) \approx F_\nu(0)/2$, or

$$\left.\frac{dJ_\nu}{d\tau_\nu}\right|_0 = \frac{3}{4}F_\nu(0) = \frac{3}{2}J_\nu(0) = \frac{3}{2}(a+c) = -c\sqrt{3\mathcal{L}_\nu} + \frac{b}{1+\eta_\nu}.$$

Therefore

$$c = \left[\frac{b}{1+\eta_\nu} - \frac{3}{2}a\right]\left(\sqrt{3\mathcal{L}_\nu} + \frac{3}{2}\right)^{-1},$$

$$J_\nu(\tau_\nu) = a + \frac{b\tau_\nu}{1+\eta_\nu} + \frac{\frac{b}{1+\eta_\nu} - \frac{3}{2}a}{\sqrt{3\mathcal{L}_\nu} + \frac{3}{2}}e^{-\sqrt{3\mathcal{L}_\nu}\tau_\nu}.$$

In the continuum, $\eta_\nu = 0, \mathcal{L}_\nu = 1$. The residual flux is then

$$r_\nu = \frac{F_\nu(0)}{F_c(0)} = \frac{J_\nu(0)}{J_c(0)} = \frac{\left(\frac{b}{1+\eta_\nu} + a\sqrt{3\mathcal{L}_\nu}\right)\left(\sqrt{3} + \frac{3}{2}\right)}{\left(b + a\sqrt{3}\right)\left(\sqrt{3\mathcal{L}_\nu} + \frac{3}{2}\right)}.$$

The residual intensity requires a specification of the source function,

$$S_\nu(\tau_\nu) = \mathcal{L}_\nu B_\nu(\tau_\nu) + (1-\mathcal{L}_\nu)J_\nu(\tau_\nu).$$

In the continuum $\mathcal{L} = 0$, so $S(\tau) = B_\nu(\tau)$.

$$f_\nu(\mu) = \frac{I_\nu(\mu,0)}{I_c(\mu,0)} = \frac{\int_0^\infty \frac{S_\nu(t_\nu)e^{-t_\nu/\mu}dt_\nu}{\mu}}{\int_0^\infty \frac{S_c(t)e^{-t/\mu}dt}{\mu}} =$$
$$\frac{a + \frac{b\mu}{1+\eta_\nu}}{a+b\mu} - \frac{(1-\mathcal{L}_\nu)\left[\frac{3}{2}a - \frac{b}{3(1+\eta_\nu)}\right]}{(a+b\mu)\left(1+\mu\sqrt{3\mathcal{L}_\nu}\right)\left(\frac{3}{2} + \sqrt{3\mathcal{L}_\nu}\right)}. \quad (67)$$

Note that the second term will vanish in the case of pure absorption ($\mathcal{L} \to 1$). In this case

$$r_\nu = \frac{a\sqrt{3} + \frac{b}{1+\eta_\nu}}{a\sqrt{3} + b}, \qquad f_\nu(\mu) = \frac{a + \frac{b\mu}{1+\eta_\nu}}{a+b\mu}. \quad (68)$$

In an isothermal atmosphere, $b = 0$ and $r_\nu = f_\nu(\mu) = 1$, and the line disappears. In the absence of temperature gradients, there can be no spectral absoprtion lines. Thus, the stronger the source function gradient, the stronger the line. Therefore, late-type stars have stronger features than early-type stars. Late-type stars have visible features at wavelengths shorter than the peak energies, where the spectrum is decaying exponentially. Early-type stars have features at wavelengths longer than the peak energies, on the Rayleigh-Jeans tail where the source function varies more slowly with temperature.

For strong absorption, $\eta_\nu >> 1$, we have

$$r_\nu = \frac{a\sqrt{3}}{a\sqrt{3}+b}, \qquad f_\nu\left(\mu\right) = \frac{a}{a+b\mu}. \qquad \eta_\nu >> 1$$

Even the strongest line vanishes as $\mu \to 0$ at the limb. For this line of sight, grazing the limb, the effects of temperature gradients are minimized. For weak absorption, $\eta_\nu << 1$, we have

$$r_\nu = 1 - \frac{b\eta_\nu}{a\sqrt{3}+b}, \qquad f_\nu\left(\mu\right) = 1 - \frac{b\mu\eta_\nu}{a+b\mu}. \qquad \eta_\nu << 1$$

The line strength is proportional to η_ν, κ_ν, and the number of absorbers.

Now consider the case of pure scattering, $\epsilon_\nu = 0$, which requires $\mathcal{L}_\nu = (1+\eta_\nu)^{-1}$ as $\kappa_\nu = 0$.

For strong scattering,

$$r_\nu = \frac{\left(\sqrt{3}+2\right)\sqrt{\mathcal{L}_\nu}}{a\sqrt{3}+b}, \qquad f_\nu\left(\mu\right) = \frac{a\sqrt{3\mathcal{L}_\nu}\left(\mu+\frac{2}{3}\right)}{a+b\mu}. \qquad \eta_\nu >> 1 \qquad (69)$$

Even if the atmosphere is isothermal, lines will persist to the limb, where the residual intensity is still about 1/2 of the residual flux. These lines do not depend upon the thermodynamic property of the gas, but upon the existence of a boundary, which permits the selective escape of photons.

For weak scattering, $\eta_\nu \to 0$ and $\mathcal{L}_\nu \to 1$:

$$r_\nu = 1 - \frac{b\eta_\nu}{a\sqrt{3}+b}$$

$$f_\nu\left(\mu\right) = 1 - \frac{b\mu\eta_\nu}{a+b\mu} - \frac{\eta_\nu\left(\frac{3}{2}a-b\right)}{\left(a+b\mu\right)\left(1+\mu\sqrt{3}\right)\left(\sqrt{3}+\frac{1}{2}\right)}. \qquad \eta_\nu << 1 \qquad (70)$$

The residual flux has the same form as for weak pure absorption, but the residual intensity is diffferent: even for an isothermal atmosphere, a weak scattering line will be visible at the limb.

The Curve of Growth of the Equivalent Width

Spectral lines are broadened from the transition frequency for a number of reasons. Thermal motions and turbulence introduce Doppler shifts between atoms and the radiation field. The probability that an atom will have a velocity v is

$$\frac{dN}{N} = \frac{e^{-(v/v_0)^2}}{\sqrt{\pi} v_0} dv,$$

where v_0 is the mean velocity (of the combined thermal and turbulent motions). The frequency ν' at which an atom will absorb in terms of the rest frequency ν_0 is

$$\nu' = \nu_0 + \frac{\nu_0 v}{c}.$$

In addition, viewed either classicaly or quantum mechanically, each transition has a damping profile or Lorentz profile, such that the atomic absorption coefficient will be proportional to

$$S_\omega \propto \frac{\Gamma_{ik}}{(\omega - \omega_0)^2 + (\Gamma_{ik}/2)^2}.$$

Here Γ_{ik} is related to the Einstein coefficient or strength of spontaneous emission, and ω_0 is the difference in energy of the states. The source of broadening in this case is due to the Heisenberg uncertainty principle. The combined effects in the atomic absorption coefficient are

$$S_\nu (v) \propto \frac{\Gamma_{ik}}{\left[\nu_0 (1 + v/c) - \nu\right]^2 + (\Gamma_{ik}/4\pi)^2}.$$

Multiplying this by the probability for the velocity and integrating over all velocities results in

$$S_\nu \propto \Gamma_{ik} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi} \nu_0} \frac{e^{-(v/v_0)^2} dv}{\left[\nu_0 (1 + v/c) - \nu\right]^2 + (\Gamma_{ik}/4\pi)^2}.$$

Define the dimensionless variables

$$u = \frac{c(\nu - \nu_0)}{\nu_0 v_0}, \qquad y = \frac{v}{v_0}, \qquad a = \frac{c\Gamma_{ik}}{4\pi \nu_0 v_0}.$$

$$S_\nu (u) \propto \frac{a}{\nu_0 v_0} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{a^2 + (u - y)^2} = \frac{\sqrt{\pi}}{\nu_0 v_0} H(a, u).$$

Here the Voigt function is

$$H\left(a,u\right) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{a^2 + \left(u - y\right)^2}.$$

There are two limiting cases that can be observed. First, for small a and small u, the Voigt function behaves as $H(a,u) \to e^{-u^2}$ since the integrand peaks at $y = u$. The opposite $u \to \infty$ limit of the Voigt function is

$$H\left(a,u\right) \to \frac{a}{\pi}\int_{-\infty}^{\infty} u^{-2}e^{-y^2}dy = \frac{au^{-2}}{\sqrt{\pi}} \qquad u \to \infty,$$

which is valid for any a.

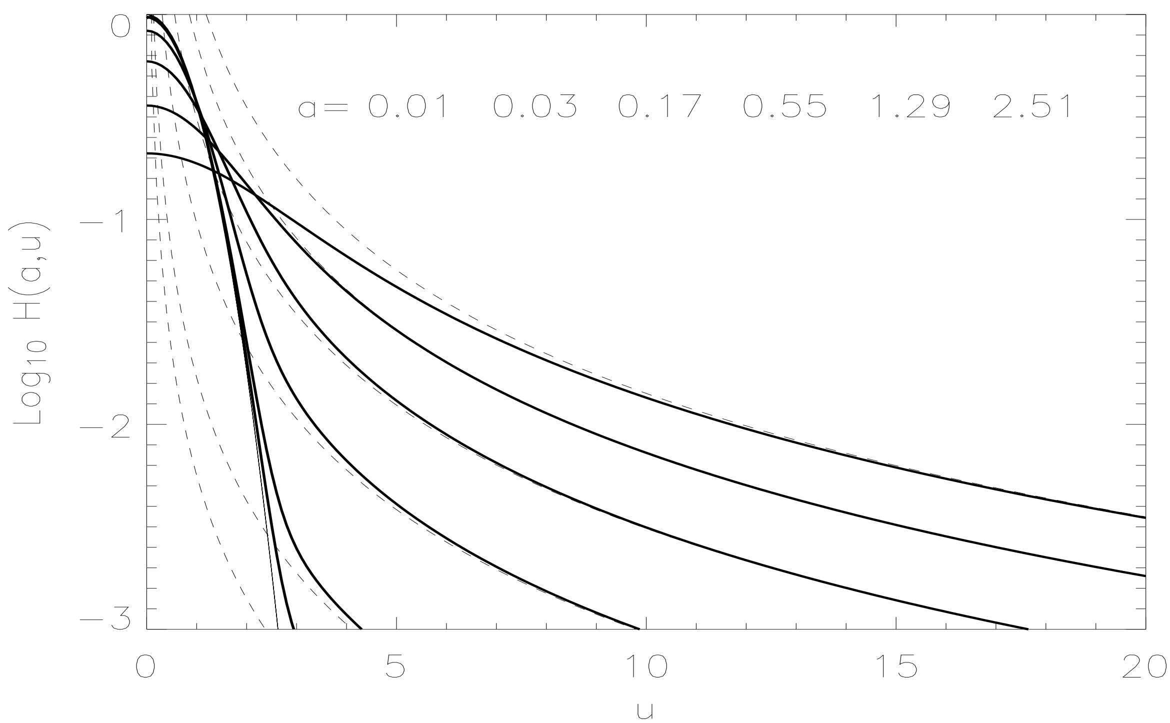


Figure 1: The Voigt function $H(a,u)$ for selected values of $a = 0.01, , 0.5, 1., 1.5, 2., 2.5, 3.$ Light solid line indicates the approximation $H(a,u) = e^{-u^2}$ appropriate for $a \approx 0$. Light dashed lines indicate the approximation $H(a,u) = a/(\pi u^2)$ appropriate for large values of u.

We can relate the size and shape of the spectral line to the abundance of the species responsible for it. Consider the Schuster-Schwarzschild model, that of a gas layer above the normal atmosphere. In this model, we have

$$r_\nu = \frac{F_\nu}{F_c} = \left(1 + \frac{\sqrt{3}\tau_0}{2}\right)^{-1},$$

where

$$\tau_0 = \int_0^{\tau_0} dt_\nu = \int_0^{z_0} \kappa_\nu \rho dz = \int_0^{z_0} n_i S_\nu dz = < S_\nu > \int_0^{z_0} n_i dz = N_i < S_\nu > .$$

N_i is the column density of the atom giving rise to the line, and $< S_\nu >$ is the line absorption coefficient averaged over depth. Neglecting depth dependences in this model we write $< S_\nu >= S_0 H(a,u)$ and $\chi_0 = S_0 N_i H(a,0)$; the line profile is

$$r_\nu = \left(1 + \frac{\sqrt{3}\tau_0}{2}\right)^{-1} = \left(1 + \frac{\sqrt{3} S_0 H\left(a,u\right) N_i}{2}\right)^{-1} = \left(1 + \frac{\sqrt{3}}{2}\chi_0 \frac{H\left(a,u\right)}{H\left(a,0\right)}\right)^{-1} .$$

The residual flux is illustrated in Fig. 2. Note that for $\chi_0 < 30$, the absorption line is not saturated. For $\chi_0 > 1000$, absorption in the wings of the line is important.

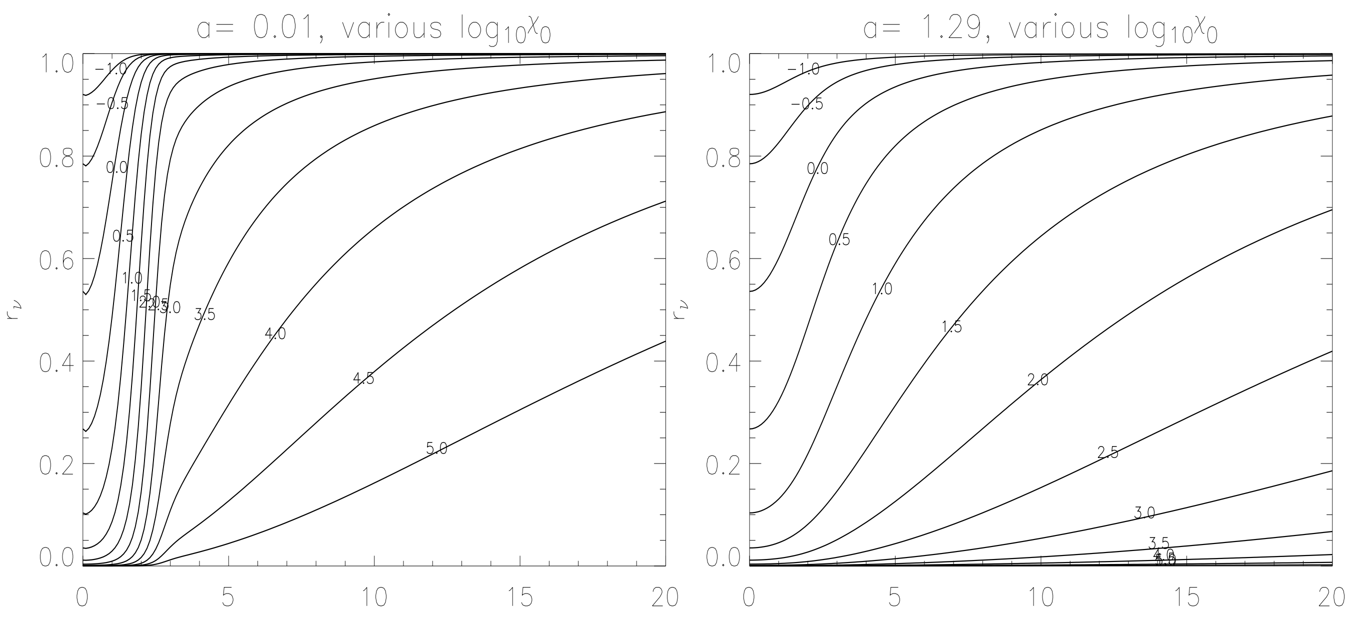


Figure 2: Residual flux in the Schuster-Schwarzschild model for an absorption line. Curves are labelled by their values of $\log_{10}\chi_0$. Two values of a, 0.01 and 1.29, are illustrated.

Now we can form the equivalent width

$$W_\lambda = \int_{-\infty}^{\infty} \frac{\sqrt{3}\tau_0/2}{1+\sqrt{3}\tau_0/2} d\lambda = 2\Delta\lambda_d \int_0^\infty \frac{\sqrt{3}S_0 H\left(a,u\right) N_i du/2}{1+\sqrt{3}S_0 H\left(a,u\right) N_i/2},$$

where $du = d\lambda/\Delta\lambda_d$. Using χ_0,

$$\frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} = \frac{2\chi_0 H\left(a,0\right)}{\sqrt{\pi}} \int_0^\infty \frac{H\left(a,u\right) du}{H\left(a,0\right)+\sqrt{3}\chi_0 H\left(a,u\right)/2}.$$

The equivalent width is shown in Fig. 3, as normalized in the above expression, for various values of a.

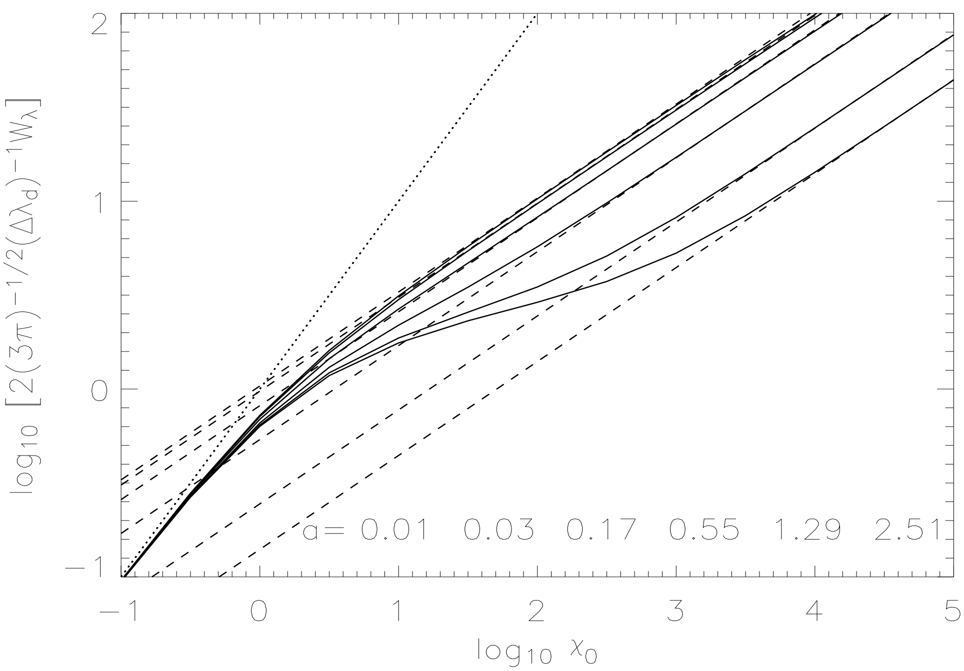


Figure 3: Equivalent widths as a function of a. The diagonal dotted line represents the small χ_0 limit, while the dashed lines illustrate the limiting behavior for $\chi_0 \to \infty$.

In the limit of weak lines ($a < 1$, moderate χ_0), using $H(a,u) \approx e^{-u^2}$ yields

$$\begin{aligned} \frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} &\simeq \frac{4H\left(a,0\right)}{\sqrt{3\pi}} \int_0^\infty du \left(1+\frac{2H\left(a,0\right)}{\sqrt{3}\chi_0} e^{u^2}\right)^{-1} \\ &= \frac{2H\left(a,0\right)}{\sqrt{3\pi}} \int_0^\infty \frac{dx x^{-1/2}}{1+e^{x-\ln\left(\sqrt{3}\chi_0/2H(a,0)\right)}} \qquad (71) \\ &= \frac{2H\left(a,0\right)}{\sqrt{3\pi}} F_{-1/2}\left(\ln\sqrt{3}\chi_0/2H\left(a,0\right)\right), \end{aligned}$$

with F the usual Fermi integral. In the limit that $\chi_0 \to 0$, this becomes

$$\frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} \simeq \chi_0 + \ldots \qquad \chi_0 << 1, \quad a < 1.$$

The equivalent width is proportional to N_i, the column density of absorbers. When χ_0 is moderately large, the opposite expansion of $F_{-1/2}$ yields

$$\frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} \simeq \sqrt{\ln\left(\frac{\sqrt{3}\chi_0}{2H\,(a,0)}\right)} + \ldots \qquad \chi_0 > 1, \quad a < 1$$

and the line saturates, increasing only as $\sqrt{\ln N_i}$. Note from Fig. 3, that for $a > 1/2$ this intermediate limit is never achieved in practice.

As the number of absorbers grows still further, however, absorption in the wings becomes important. The relevant case is to take the large u limit of $H(a,u) \propto u^{-2}$. W_λ will thus grow faster again:

$$\frac{2}{\sqrt{3\pi}}\frac{W_\lambda}{\Delta\lambda_d} \simeq \frac{4H\,(a,0)}{\sqrt{3\pi}} \int_0^\infty du \left(\sqrt{\frac{\pi}{3}}\frac{2H\,(a,0)}{a\chi_0}u^2 + 1\right)^{-1}$$
$$= \left(\frac{\pi}{3}\right)^{1/4} \sqrt{2aH\,(a,0)\,\chi_0}, \qquad \chi_0 >> 1$$

which depends on $\sqrt{N_i}$.

These results are valid for scattering lines, but other applications may require a more sophisticated treatment.