Homework 5 Solutions

1. Derive the homologous form of the luminosity equation, \( \frac{df}{dx} = Dp^2t^\nu x^2 \).

The luminosity equation is \( \frac{dL}{dr} = 4\pi r^2 \rho(r) \epsilon(r) \). You also need to use the fact that \( \epsilon(r) = \epsilon_0 \rho T^\nu(r) \).

Define the homologous variables \( f = \frac{L(r)}{L} \), \( p = \frac{P(r)}{P_0} \), \( t = \frac{T(r)}{T_0} \), \( x = \frac{r}{R} \). We showed in class that the central values are \( P_0 = \frac{GM^2}{4\pi R^2} \) and \( T_0 = \frac{\mu GM}{R \nu} \).

The homologous luminosity relation is \( \frac{df}{dx} \).

Use \( df = \frac{dL(r)}{L} \) and \( dx = \frac{dr}{R} \) to get \( \frac{df}{dx} = \frac{R}{L} \frac{dL(r)}{dr} = \frac{R}{L} \frac{4\pi r^2 \rho(r) \epsilon(r)}{R} = \frac{R}{L} \frac{4\pi x^2 R^2 \rho(r) \epsilon_0(r) \rho T^\nu(r)}{R} \).

To get the homologous form of the density, use the gas equation, \( \rho = \frac{\mu}{R \nu T} P \), and use the homologous equation for the pressure.

Plugging in, \( \frac{df}{dx} = \frac{R^3}{L} \frac{4\pi x^2 \epsilon_0 T^\nu \left( \frac{\mu}{R \nu} \right)^2 \left( \frac{P}{T_0} \right)^2}{P} \).

Collecting terms, this simplifies to \( \frac{df}{dx} = \frac{R^3}{L} \frac{4\pi x^2 \epsilon_0 \left( \frac{\mu GM}{R \nu} \right)^2 \nu^{-2} x^{-2} \left( \frac{\mu}{R \nu} \right)^2 \left( \frac{GM^2}{4\pi R^2} \right)^2}{P^2 \nu^{-2} x^2} = \frac{4\pi x^2 \epsilon_0 ^2 \mu^2 M^4}{4\pi R^4} \frac{GM^2}{R^2} \).

Collect the two fractions into \( D \) (note that \( \frac{G^\nu \epsilon_0}{4\pi R^2} = D_0 \)). Therefore, \( \frac{df}{dx} = \frac{Dp^2t^\nu x^2}{R^4} \).

2. C&O 11.2.
   (a.) The mass loss is \( M = \frac{L}{c^2} \) = 6.8 \times 10^{-14} M_\odot/year (this is about 10^6 tons/second).
   (b.) The solar wind mass loss rate is about 3 \times 10^{-14} M_\odot/year (see example 11.2.1), or half the nuclear in.
   (c.) Over the \( \sim 10^{10} \) year lifetime of the Sun, it will lose about \( 10^{-3} \) M_\odot from these 2 processes. This is small - we cannot generally measure stellar masses to this accuracy. To some extent this is offset by comets and asteroids falling into the Sun.

3. C&O 11.5
   (a.) Let \( T = 5780 \) K, \( m = m_H \), and \( \lambda = 6563 \) Å. \( v_{\text{turb}} = 0 \). Plug in to get \( \lambda_{\nu} = 0.036 \) nm = 0.36 Å.
   (b.) \( v_{\text{turb}} \approx 0.4 \) km/s (see pg. 364 of your text). Plug this into equation 9.63 to get \( \Delta \lambda_{\nu} = 0.036 \) nm. Turbulence makes very little difference.
   (c.) The answer to (b.) makes sense because \( \frac{v_{\text{turb}}}{2kT/m} \approx 0.002 \).
   (d.) This is not a well-worded problem. In the photosphere, (part a.), the answer is clearly no for Hydrogen. Heavier elements will have the same \( v_{\text{turb}} \) but smaller thermal velocities \( \approx \frac{1}{m} \). Lines formed higher up in the atmosphere, in the chromosphere or corona, have higher thermal velocities. Turbulence is most important for heavy elements formed in the photosphere, like Fe I.
4. C&O 12.10
   Let T = 10K, \( \mu=1 \), and \( \rho_0=3\times10^{-17} \text{ kg/m}^3 \). Use equation 12.16 to get \( R_f = 4.5\times10^{15} \text{ m} \approx 0.5 \text{ ly} \).

5. C&O 12.18
   (a.) Start with equation 12.19. Adding the centripetal term gives 
   \( \frac{d^2r}{dt^2} = v_r \frac{dv_r}{dr} = -\frac{GM_r}{r^2} + \omega^2 r \). The product \( I\omega \) is conserved, so \( I\omega = I_0\omega_0 \). The moment of inertia \( I \) for rotation, not a Keplerian orbit.
   (b.) \( \omega_0 = \sqrt{2GM_r/r_0^2} = 2.7\times10^{-16} \text{ rad/s} \)
   (c.) \( I_0\omega_0 = I_f\omega_f \), with \( I_0 = \frac{2}{5}MR^2 \) and \( I_f = \frac{1}{2}MR^2 \). Plug in to get \( \omega_f = 2.3\times10^{-10} \text{ rad/s} \), and \( v_f = 3.4 \text{ km/s} \).
   (e.) The rotation period \( P = 2\pi r_f/v_f = 2.8\times10^{10} \text{ sec} = 880 \text{ years} \).

6. T Tauri stars collapse quasi-statically, converting gravitational potential energy into heat. If you could measure the radius of a T Tauri star very accurately, how much smaller would it be after a year? Assume a mass of \( 1 M_\odot \), a radius of \( 3 R_\odot \), and \( T_{eff}=4500\text{K} \). If you had a photometer capable of measuring the brightness with 0.001 magnitude accuracy, how long would it take before you noticed the star fading (ignore the fact that all T Tauri stars are stochastically variable on timescales of days to years)?

   The internal energy of the star is \( \frac{3}{5}GM_r^2 \). The energy radiated away, \( \Delta E \), is approximated by \( \Delta E = -\frac{3}{5}GM_r^2 \left( \frac{1}{R} - \frac{1}{R-\Delta R} \right) \), where \( \Delta R \) is the change in radius.
   \[
   \frac{1}{R-\Delta R} = \frac{\Delta R}{R^2-\Delta R^2}. \quad \Delta R << R, \text{ so } \Delta E \approx -\frac{3}{5}GM_r^2 \frac{\Delta R}{R^2}.
   \]
   \( L = \frac{\Delta E}{\Delta t} \). Use the approximation that \( L = 4\pi R^2 \sigma T^4 \) to get \( \frac{\Delta R}{R} = -\frac{5}{3} \frac{4\pi R^4 \sigma T^4}{GM_r^2} \approx 0.004 \text{ cm/s} \). In one year the star contracts by \( 1.1\times10^5 \text{ cm} \).

   Note that the contraction rate, \( \frac{\Delta R}{R} \approx 5 \times 10^{-7} \), is the inverse of the contraction timescale, which is about \( 2 \times 10^6 \text{ years} \). This is the roughly the Kelvin-Helmholtz timescale.

   Assume that the radius changes at constant temperature. For a change in radius \( \delta \), \( \Delta \text{mag} = 0.001 = 2.5 \log\left(\frac{L'}{L}\right) = 2.5 \log\left(\frac{R-\delta}{R}\right)^2 = 5 \log\left(\frac{R-\delta}{R}\right) \). Solve to find \( \delta = 4.6\times10^{-4}R \). Using the contraction rate you calculated earlier, this would take about 900 years.
No, you cannot observe this in a human lifetime.