AST 341
Final Exam and Solutions

18 December 2008

Please put your answers in a Blue Book. Show all your work - partial credit will be given. Think before writing. Please be neat. If you decide to change your answer, neatly cross out the incorrect answer (no need to completely redact it) and start over. Consider using the last page or two of the book as scratch space.

Points may not be given for illegible answers.

All questions (or parts thereof) are worth 5 points, except as noted.

Calculators are not necessary, and may not be used.

No answer should require more than one page; all can be answered in a few sentences or lines. In general the longer your answer, the lesser your understanding.

You have the full 150 minute exam period, but you should be able to complete the exam in half that time.

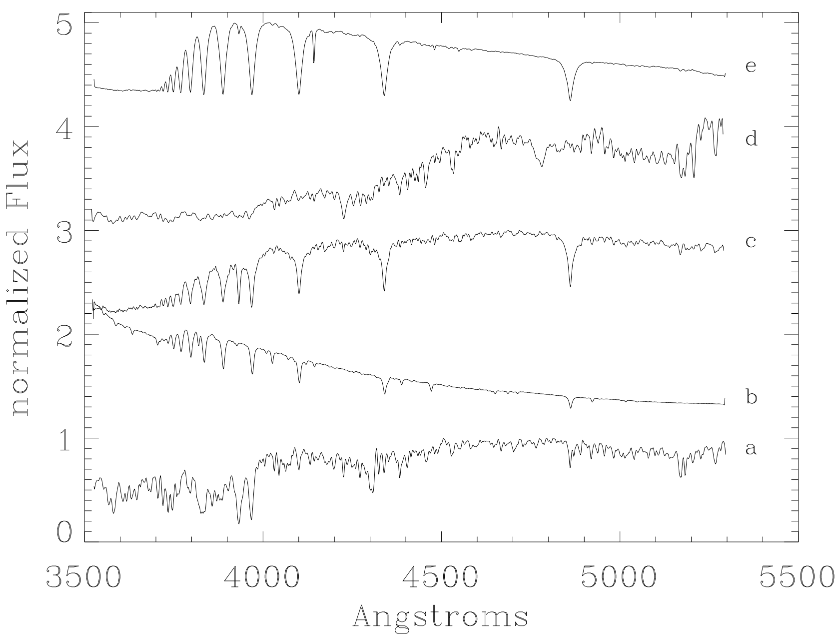
Useful Numbers and Formulae

c=2.99792×10^{10} cm s^{-1}	G=6.672×10^{-8} cm^3 gm^{-1} s^{-2}
h=6.626×10^{-27} erg s	$\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} K^{-4}
k= 1.38×10^{-16} erg K^{-1}	R$_g$ = 8.314×10^7 erg K^{-1} mole^{-1}
L$_\odot$=4×10^{33} erg s^{-1}	M$_\odot$=2×10^{33} gm
$m_H = 1.67 \times 10^{-24}$ gm	$\sigma_T = 6.65 \times 10^{-25}$ cm^2
1 AU = 1.5×10^8 km	eV = 1.602×10^{-9} erg
P = nkT	$\frac{N_i N_e}{N_0} = \frac{2U_1(T)}{U_0(T)}(\frac{2\pi mk}{h^2})^{\frac{3}{2}} T^{\frac{3}{2}} e^{-\frac{\chi_0}{kT}}$
L=$4\pi R^2 \sigma T^4$	$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{(e^{\frac{h\nu}{kT}}-1)}$
$\Delta x \Delta p_x > h$	

1. *State the Russell-Vogt theorem.*

   The properties of a star are solely determined by its mass, age, and composition. Or, given an initial mass and composition, the history of a star is uniquely determined. To first order this assumption holds. Stellar rotation and magnetism have subtle effects.

2. *The figure shows spectra of 5 main sequence stars. Place these in a temperature sequence, with the hottest star first. Explain your reasoning.*



   The hottest star is **b**. It has the bluest continuum, and is a B0 star (the weak lines are mostly Helium). Star **e** (A0) has the strongest Balmer lines. It is hotter than star **c** (F0); it has a bluer continuum and more metallic lines between the Hydrogen lines. Star **a** (G0) has only weak Balmer lines. The coolest is star **d** (M0), which has very little blue flux.

   I failed to mention that the spectra are offset from one-another by one unit of flux; that may have mislead some of you into thinking that, for example, star c has a lot of blue flux.

3. *Why is there a main sequence?*

   Stars are stable, and are in hydrostatic equilibrium while they undergo core Hydrogen burning. This phase lasts about 90% of a star's life, therefore most stars will be found in this phase. The Russell-Vogt theorem shows that L should depend on M, hence a unique locus for most of the stars in the H-R diagram.

4. *Show that the lifetime τ of a star on the main sequence is roughly given by $\tau = m^{-2}$, where m is the stellar mass.*

The timetime τ is equal to the amount of energy generated divided by the energy radiation rate. The amount of energy is proportional to the stellar mass. The radiation rate is the luminosity. On the main sequence, L$\propto$M^3, so $\tau \propto$ M/L $\sim$ M^{-2}.

5. *Use the concept of hydrostatic equilibrium (pressure balance) to show that the central pressure P_c in a star scales as $\frac{M^2}{R^4}$ (10 points).*

   In hydrostatic equilibrium the pressure balance is given by $\frac{dP}{dr} = \frac{Gm(r)\rho(r)}{r^2}$, where the right hand side is the mass loading. Assume a mean density $\rho = \frac{M}{r^3}$. You can either integrate this equation with the boundary conditions that the pressure at the surface is zero and M,R are the mass and radius of the star, or do a dimensional analysis. $\frac{dP}{dr} \sim \frac{P}{R} = \frac{GM\frac{M}{R^3}}{R^2} \sim \frac{M^2}{R^5}$, so $P \sim \frac{M^2}{R^4}$.

6. *Starting with $P_c \propto \frac{M^2}{R^4}$, show that the temperature T_c in the stellar core scales as $T_c \propto \frac{M}{R}$ (10 points).*

   The internal pressure is provided by thermal gas, $P_g = nkT = \frac{\rho kT}{m_H}$. Let $\rho = \frac{M}{R^3}$. $\frac{M^2}{R^4} \sim \frac{MT}{R^3}$, hence $T \sim \frac{M}{R}$.

7. *What is the principle nuclear reaction that occurs in the Sun? Use the fact that the mass deficit (the amount of mass converted to energy) is 0.007 to determine the main sequence lifetime of the Sun. Show your work, and clearly state all assumptions. This is an estimation problem: do it without a calculator to get an order of magnitude estimate. (10 points)*

   There are a number of ways to do this problem. One is to realize that $L = 0.007mc^2$ gives the mass of Hydrogen being converted to Helium every second. The amount of mass available is ~ 0.1M$_\odot$ (the mass in the core). The lifetime is then $\tau \approx 0.1\mathrm{M}_\odot / \frac{L}{0.007c^2}$ $= 0.1\times2\times10^{33}0.007\times9\times10^{20}/4\times10^{33}$ seconds $\approx \frac{2\times7\times9}{4}\times10^{(33-1-3+20-33)} \approx 14\times10^{16} =$ 1.5×10^{17} seconds. There are $\pi\times10^7$ seconds in a year. The solar lifetime is roughly 5×10^9 years, which is within a factor of 2 of the accepted value.

8. *What is the main source of photospheric opacity in*

   (a) *the Sun?* H^- ions

   (b) *an M5 star?* molecules (mostly TiO and VO)

   (c) *an O star?* electron scattering (the gas is nearly fully ionized)

9. *Why does the luminosity of the Sun increase by about a factor of 2 while on the main sequence? (10 points).*

   As you convert 4 protons into 1 Helium nucleus, you remove particles from the core. P=nkT. P must offset the stellar mass (see question 5), and cannot change. Since n decreates, T must increase to compensate. Nuclear reaction rates scale as T$^\nu$, where $\nu > 0$, so the energy generation, and luminosity, must increase.

10. *Derive the Eddington luminosity by equating the radiation pressure on electrons with the gravitational force on the proton. With the assumption that L$\propto$M^3 and R$\propto$M on*

the upper main sequence, estimate the maximum mass of a main sequence star. (10 points)

The pressure balance at the stellar radius R is between $F_g = \frac{GMm_p}{r^2}$ and $F_{rad} = \frac{L}{4\pi r^2 c}\sigma_T$. The distance r cancels. The pressures balance when $L_{Edd} = \frac{4\pi cGMm_p}{\sigma_T}$, or $M = \frac{L\sigma_T}{4\pi cGm_p}$.

To see where this affects main sequence stars, assume $L = L_\odot(\frac{M}{M_\odot})^3$ (the M-L relation on the main sequence). $M_{max} = \frac{L_\odot(\frac{M}{M_\odot})^3\sigma_T}{4\pi cGm_p}$m or $M^2_{max} = \frac{4\pi cGm_p M^3_\odot}{L_\odot\sigma_T}$

$M^2_{max} \approx \frac{123\times10^{10}6.67\times10^{-8}1.67\times10^{-24}8\times10^{99}}{4\times10^{33}6.6\times10^{-25}} \approx \frac{12\times3\times8\times6.67\times1.67}{4\times6.65} \times 10^{10-8-24+99-33+25} \approx 120 \times 10^{69}\text{gm}^2$. $M_{max} \approx 3 \times 10^{35}$gm, or about 150 $M_\odot$.

11. *The probability that two atoms can approach close enough for the strong force to initiate a nuclear reaction in the Solar core is small. The electrostatic potential $U{=}k\frac{Z_1Z_2e^2}{r}$ where Z_1 and Z_2 are the charges on the two particles, e is the electric charge, and r is the distance between the particles. k is one in cgs units, i.e., when the charge is expressed in electrostatic units ($e = 4.8 \times 10^{-10}$ ESU) and the distance is in cm.*

   (a) *Estimate the temperature at which the thermal energy of a proton is equal to this potential energy (let $Z_1 = Z_2 = 1$).*
   Let $\frac{3}{2}kT = \frac{Z_1Z_2e^2}{r}$ and solve for T. $T = \frac{2}{3}\frac{e^2}{r}$, where r is the size of the nucleus (10^{-13} cm) because the particles must approact this close to interact. I gave you the wrong value for e in ESU, but it didn't really matter.
   $T \approx 0.6 \times 25/1.38 \times 10^{-20+13+16} \approx 1.2 \times 10^9$K.

   (b) *How does this compare to the core temperature in the Sun?*
   The solar core is about $1.5{\times}10^7$K - much cooler

   (c) *What implications does this have for how nuclear reactions actually proceed occur in the Sun?*
   Nuclear reaction require quantum mechanical tunneling.

   (d) *Estimate (*do not calculate*) from general principles about how long it will take a typical proton in the Solar core to fuse.*
   Since it takes 10^{10} years to burn up all the Hydrogen, the typical proton will react in about half that time, of $5{\times}10^9$ years.

12. *High mass stars burn elements up to Fe^{56}.*

   (a) *Why do they stop there?*
   Fe^{56} has the highest binding energy of any nucleus - reactions that build up heavier elements are endothermic in nature, and do not release energy.

   (b) *How are heavier elements produced?*
   Mostly by r-process neutron capture in supernovae, but also by the s-process (up to Bi^{209} in red giants).

13. *Show that degeneracy pressure scales as $n_e^{5/3}$. The pressure due to collisions is given by $P = nvp$ where n, v, and p are respectively the number density, velocity, and momenta of the colliding particles. Let the mean electron density be n_e.*

(a) *What is the typical distance between electrons?*

$n_e^{-\frac{1}{3}}$

(b) *Use the uncertainty principle and your answer to part (a) to show that the momentum is of order $hn_e^{1/3}$.*

$\Delta x \Delta p \approx h$ or $p \approx hn_e^{\frac{1}{3}}$

(c) *What is the velocity of a non-relativistic electron?*

$v = p/m_e = \frac{hn_e^{\frac{1}{3}}}{m_e}$

(d) *Use your answer to (c) to show that the degenerate pressure $P_e \sim \frac{h^2 n_e^{5/3}}{m_e}$*

$P_e = npv \sim n_e h n_e^{\frac{1}{3}} \frac{hn_e^{\frac{1}{3}}}{m_e} = \frac{h^2 n_e^{5/3}}{m_e}$

(e) *What is n_e in terms of the mass density ρ and the mass of the proton m_p? For simplicity, assume star is composed entirely of protons and electrons, so the charge-to-mass ratio $\frac{Z}{A} = 1$. Assume charge neutrality (equal numbers of protons and electrons). Rewrite P_e in terms of ρ rather than n_e.*

$n_e = \frac{\rho}{m_p}$

(f) *You should have found that $P_e \sim \rho^{5/3}$. Use this to derive the mass-radius relation for white dwarfs. How does if differ from that for main-sequence stars, where gas pressure provides the support?*

$P_e \sim \rho^{5/3} \sim \frac{M^2}{R^4}$. Plug in $\rho \sim \frac{M}{R^3}$ to get $R \sim M^{-\frac{1}{3}}$. The more massive the star, the smaller its radius. Is is the opposite to the behavior on the main sequence, where the radius increases with increasing mass.

14. *What is the shortest orbital period a pair of identical main sequence stars can have? Express your answer in terms of the mass in solar units. Assume that $R \propto M$. (10 points)*

(Note: I left out the word "identical", which would result is a better-defined problem.)

Use Newton's formulation of Kepler's 3^{rd} law. $P^2 = \frac{4\pi^2}{G} a^3 (M_1 + M_2)$. Let $M_1 = M_2$. The minimum separation $a_{min} = R = \frac{M}{M_\odot} R_\odot$

Plug in, to get $P^2 = \frac{7^3 \times 10^{30} M^3 4 \times 10}{6.67 \times 10^{-8} 2 \times 2 \times 10^{33}} \sim \frac{50 \times 7 \times 4}{6.67 \times 4} \times 10^{30+1-33+8} \sim 5 \times 10^7 \frac{M}{M_\odot}$ sec, or about 2 hours for a pair of solar mass stars.

15. *The sound speed v_s in an isothermal medium is given by $v_s^2 = \gamma \frac{P}{\rho}$. Use the ideal gas law to argue that the fundamental pulse period of a star is proportional to $\sqrt{(R^3/M)}$. From this, derive the Cepheid period-luminosity, $log(f) \sim \alpha M_V$, where f is the period, M_V is the absolute visual magnitude, and α is a proportionality constant. Estimate the value of α . (10 points)*

There are a number of ways to approach this. One is to use $P = \rho kT$ to get $v_s^2 \propto T \sim \frac{M}{R}$. The fundamental pulsation period is $f = \frac{2R}{V} \sim \frac{2R}{\sqrt{M/R}} \sim \frac{2}{\sqrt{M/R^3}} \sim \frac{1}{\sqrt{\rho}}$.

Approximate the star as a blackbody. $L \sim R^2$ (the temperature range is small, so the T^4 dependence is less important). $L \sim R^2 \sim f^{\frac{4}{3}}$, or $\log f \sim \frac{3}{4} logL$. The definition of magnitude is $L = 2.5^{-M}$, so $\log f \sim -0.4\frac{3}{4}M \sim= -0.3M$. Because the range in temperature is small, the change in bolometric correction is small, and you can replace M with M_V.

16. *The solar corona is fully ionized. The mean electron density* $n_e \sim 10^9\ cm^{-3}$. *The temperature is about* $2 \times 10^6 K$. *Assume that the coronal radius is approximately equal to the solar radius. What is the approximate optical depth of the corona to electron scattering?*

    The optical depthe $\tau = nl\sigma$, where sigma is the scattering cross section. In this case $\sigma = \sigma_T$. Plug in $n = 10^9$ cm^{-3} and $l = 7 \times 10^{10}$cm to get $\tau = 7 \times 6.6 \times 10^{10+9-25} \sim 4.5 \times 10^{-6}$. The corona is optically thin.